SYMMETRIES IN CLASSICAL FIELD THEORY

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Abstract

The multisymplectic description of Classical Field Theories is revisited, including its relation with the presymplectic formalism on the space of Cauchy data. Both descriptions allow us to give a complete scheme of classification of infinitesimal symmetries, and to obtain the corresponding conservation laws.

1 Introduction

The multisymplectic description of Classical Field Theories goes back to the end of the sixties, when it was developed by the Polish school leadered by W. Tulczyjew (see [3, 36, 37, 38, 68]), and also independently by P.L García and A. Pérez-Rendón [20, 21, 22], and H. Goldschmidt and S. Sternberg [25]. From that time, this topic has continuously deserved a lot of attention mainly after the paper [7], and more recently in [19, 33, 34, 61, 62]. A serious attempts to get a full development of the theory has been done in the monographs [28, 29] (see also [54] for higher order theories). In addition, multisymplectic setting is proving to be useful for numerical purposes [56].

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The final goal is to obtain a geometric description similar to the symplectic one for Lagrangian and Hamiltonian mechanics. Therefore, the first idea was to introduce a generalization of the symplectic form. The canonical symplectic structure on the cotangent bundle of a configuration manifold is now replaced by multisymplectic forms canonically defined on the bundles of exterior forms on the bundle configuration $\pi : Y \longrightarrow X$ of the theory in consideration. These geometric structures can be abstracted to arbitrary manifolds; its study constitutes a new subject of interest for geometers [5, 6, 52, 57, 58] which could give new insights as it happened with symplectic geometry in the sixties.

On the other hand, if we start with a Lagrangian density, we can construct first a Lagrangian form from a volume form fixed on the space-time manifold X, and then, using the bundle structure of the 1-jet prolongation $\pi_{XZ} : Z \longrightarrow X$ of Y, we construct a multisymplectic form on Z (provided that the Lagrangian is regular).

In this geometric context, one can present the field equations in two alternative ways: in terms of multivectors (see [11, 12, 13, 14, 15, 16, 17, 18, 19]), or in terms of Ehresmann connections [44, 48, 49, 52].

Let us remark that there are alternative approaches using the so-called polysymplectic structures (see [23, 24, 35, 63, 64, 65]) or even n-symplectic structures (see [53] for a recent survey). Here, we shall present the field equations in terms of Ehresmann connections; indeed, note that in Lagrangian or Hamiltonian mechanics one looks for curves, or, in an infinitesimal version, tangent vectors; now, we look for sections of the corresponding bundles, which infinitesimally correspond to the horizontal subspaces of Ehresmann connections. In fact, the Euler-Lagrange equations (more generally, teh De Donder equations) and Hamilton equations can be described in a form which is very similar to the corresponding ones in Mechanics. Both formalisms (Lagrangian and Hamiltonian) are related via the Legendre transformation. The case of singular theories is also considered, and a constraint algorithm is obtained.

Accordingly with these different descriptions, we have different notions of infinitesimal symmetries (see [60] for a description based in the calculus of variations). The aim of the present paper is to classify the different kind of infinitesimal symmetries and to study their relationship with conservation laws in the geometric context of multisymplectic geometry and Ehresmann connections.

In addition, choosing a Cauchy surface, we also develop the corresponding infinite dimensional setting in the space of Cauchy data. Both descriptions are related by means of integration along the Cauchy surfaces, allowing to relate the above symmetries with the ones of the presymplectic infinite dimensional system.

Let us remark that we consider boundary conditions along the paper.

The paper is structured as follows. Section 2 describe the Lagrangian setting for the Classical Field Theories of first order using the tools of jet manifolds, in both regular and singular cases. Multisymplectic forms and brackets are introduced at the end of the section in order to be used later. Section 3 is devoted to give a Hamiltonian description for Classical Field

Theories, including the Legendre transformation and the equivalence theorem. The singular case is also discussed. Section 4 deals with the theory of Cauchy surfaces for the Classical Field Theory, where the tools that will be required later are introduced. In particular, the integration method, as a way to connect the finite dimensional setting and the theory of Cauchy Surfaces, is discussed in depth. The singular case and the Poisson brackets are also considered. Section 5 describes thoroughly the different infinitesimal symmetries for the Lagrangian and Hamiltonian settings, using the tools that have been described in previous sections. In Section 6, we discuss the Momentum Map in the finite and infinite dimensional settings. The paper finishes with section 7, in which we illustrate the concepts discussed with the examples of the Bosonic string, following the Polyakov approach, and the Klein-Gordon field.

Along this paper, we shall use the following notations. $\mathfrak{X}(M)$ will denote the Lie algebra of vector fields on a manifold M, and $\pounds_X$ will be the Lie derivative with respect to a vector field X. The differential of a differentiable mapping $F : M \longrightarrow N$ will be indistinctly denoted by F_*, dF or TF. By $C^\infty(M)$ we denote the algebra of smooth functions on a manifold M.

2 Lagrangian formalism

2.1 The setting for classical field theories

Consider a fibration $\pi = \pi_{XY} : Y \longrightarrow X$, where Y is an $(n + 1 + m)$-dimensional manifold and X is an orientable $(n + 1)$-dimensional manifold. We shall also fix a volume form on X, that will be denoted by η. We can choose fibered coordinates (x^μ, y^i) in Y, so that $\pi(x^\mu, y^i) = (x^\mu)$, and assume that the volume form is $\eta = d^{n+1}x = dx^0 \wedge \ldots \wedge dx^n$. Here, $0 \leq \mu, \nu, ... \leq n$ and $1 \leq i, j, ... \leq m$.

Remark 2.1. *Time dependent mechanics can be considered as an example of classical field theory, where X is chosen to be the real line $\mathbb{R}$, representing time, and the fibre over t represents the configuration space at time t.*

We shall also use the following notation:

$$d^n x_\mu := \iota_{\partial/\partial x^\mu} d^{n+1}x, \quad d^{n-1}x_{\mu\nu} := \iota_{\partial/\partial x^\mu} \iota_{\partial/\partial x^\nu} d^{n+1}x, \quad \ldots$$

The first order jet prolongation $J^1\pi$ is the manifold of classes $j^1_x\phi$ of sections ϕ of π around a point x of X which have the same Taylor expansion up to order one. $J^1\pi$ can be viewed as the generalisation of the phase space of the velocities for classical mechanics. Therefore, $J^1\pi$, which we shall denote by Z, is an $(n + 1 + m + (n + 1)m)$-dimensional manifold. We also define the canonical projections $\pi_{XZ} : Z \longrightarrow X$ by $\pi_{XZ}(j^1_x\phi) = x$, and $\pi_{YZ} : Z \longrightarrow Y$ by $\pi_{YZ}(j^1_x\phi) = \phi(x)$ (see Figure 1). We shall also use the same notation η for the pullback of

the chosen volume form η on X to Z along the projection. If we have adapted coordinates (x^μ, y^i) in Y, then we can define induced coordinates in Z, given by (x^μ, y^i, z^i_μ), such that

$$
\begin{aligned}
x^\mu(j^1_x\phi) &= x^\mu(x) \\
y^i(j^1_x\phi) &= y^i(\phi(x)) = \phi^i(x) \\
z^i_\mu(j^1_x\phi) &= \frac{\partial \phi^i}{\partial x^\mu}\bigg|_x
\end{aligned}
$$

$$
\begin{tikzcd}
 & Z = J^1\pi_{XY} \arrow[dl, "\pi_{YZ}"'] \arrow[dd, "\pi_{XZ}"] & & & (x^\mu, y^i, z^i_\mu) \arrow[dl] \arrow[dd] \\
Y \arrow[dr, "\pi_{XY}"] & & & (x^\mu, y^i) \arrow[dr] & \\
 & X \arrow[ul, bend left, "\phi"] \arrow[uu, dotted, bend right, "j^1\phi"'] & & & (x^\mu)
\end{tikzcd}
$$

$$
\begin{cases}
\dim X = n+1 \\
\dim Y = n+1+m \\
\dim Z = n+1+m+(n+1)m
\end{cases}
$$

Figure 1

As usual, one can define the concept of verticality, by defining the following subbundles:

$$
\begin{aligned}
\mathcal{V}_y\pi &:= (T_y\pi)^{-1}(0_x) \\
\mathcal{V}_z\pi_{XZ} &:= (T_z\pi_{XZ})^{-1}(0_x)
\end{aligned}
$$

We can consider the more general case in which X is a manifold with boundary ∂X, and we also have boundaries for manifolds Y and Z, given by $\partial Y = \pi^{-1}(\partial X)$ and $\partial Z = \pi^{-1}_{XZ}(\partial X)$, respectively. A boundary condition is encoded in a subbundle B of $\partial Z \longrightarrow \partial X$, and restricting ourselves to sections $\phi : X \longrightarrow Y$ such that $j^1\phi(\partial X) \subseteq B$ (see [3]).

There are several other alternative (and equivalent) definitions of the first order jet bundle, such as considering the affine bundle over Y whose fibre over $y \in \pi^{-1}(x)$ consists of linear sections of $T\pi_{XY}$, modelled over the vector bundle on Y whose fibre over $y \in \pi^{-1}(x)$ is the space of linear maps of T_xX to $\mathcal{V}_y\pi$; in other words, Z is an affine bundle over Y modelled on the vector bundle $\pi T^*X \otimes_Y \mathcal{V}\pi$ (see [28, 66, 67]).

The first order jet bundle is equipped with a geometric object S_η, which depends on our choice of the volume form, called vertical endomorphism (see [9] or [67]). What follows is an

alternative way to define it. First of all, we construct the isomorphism (vertical lift)

$$\upsilon : \pi^* T^* X \otimes_Y \mathcal{V}\pi \longrightarrow \mathcal{V}\pi_{YZ}$$

as follows: given $f \in (\pi^* T^* X \otimes_Y \mathcal{V}\pi)|_{j^1\phi}$ consider the curve $\gamma_f : \mathbb{R} \longrightarrow \pi_{YZ}^{-1}(\pi_{YZ}(j^1_x\phi))$ given by

$$\gamma_f(t) = j^1_x\phi + t\ f\ ,$$

for all $t \in \mathbb{R}$. Now define

$$f^v = \frac{d}{dt}\gamma_f(t)|_{t=0}$$

If (x^μ, y^i) are fibered coordinates on Y and $f = f^i_\mu dx^\mu|_x \otimes \left.\frac{\partial}{\partial y^i}\right|_{\phi(x)}$ then

$$f^v = f^i_\mu \left.\frac{\partial}{\partial z^i_\mu}\right|_{j^1_x\phi}\ .$$

Let x be a point of X and $\phi \in \Gamma_x(\pi)$, where $\Gamma_x(\pi)$ denotes the set of all local sections around the point x. If $V_0, \ldots, V_n$ are $n+1$ tangent vectors to $J^1\pi$ at the point $j^1_x\phi \in Z$, then we have that $T_{j^1_x\phi}\pi_{YZ}(V_i) - T_x\phi \circ T_{j^1_x\phi}\pi_{XZ}(V_i) \in (\mathcal{V}\pi)_{\phi(x)}$ (this is the vertical differential of a vector field on Z). From the volume form η, we also construct a family of 1-forms η_i as follows:

$$\eta_i(x) = (-1)^{n+1-i} i_{T_{j^1_x\phi}\pi_{XZ}(V_0)} \cdots \widehat{i_{T_{j^1_x\phi}\pi_{XZ}(V_i)}} \cdots i_{T_{j^1_x\phi}\pi_{XZ}(V_n)}\ \eta(x)\ ,$$

where the hat over a term means that it is omitted.

Next, we define the **vertical endomorphism** S_η as follows:

$$(S_\eta)_{j^1_x\phi}(V_0, \ldots, V_n) = \sum_{i=0}^{n} \left(\eta_i(x) \otimes (T_{j^1_x\phi}\pi_{YZ}(V_i) - T_x\phi \circ T_{j^1_x\phi}\pi_{XZ}(V_i))\right)^v$$

Whenever we pick a different volume form $F\eta$, then $(F\eta)_i = F\eta_i$, whence we also get $S_{F\eta} = FS_\eta$, where $F : X \longrightarrow \mathbb{R}$ is nowhere-vanishing smooth function on X.

The vertical endomorphism can be also written in local induced coordinates as follows

$$S_\eta = (dy^i - z^i_\mu dx^\mu) \wedge d^n x_\nu \otimes \frac{\partial}{\partial z^i_\nu}$$

Higher order jet bundles can be defined in a similar manner. The second order jet bundle, for example, is an $(n+1+m+(n+1)m+\begin{pmatrix} n+2 \\ 2 \end{pmatrix} m)$-dimensional manifold, which has induced coordinates $(x^\mu, y^i, z^i_\mu, z^i_{\mu\nu})$, where

$$z^i_{\mu\nu}(j^1_p\phi) = \left.\frac{\partial^2\phi^i}{\partial x^\mu \partial x^\nu}\right|_p$$

These bundles allow us to define the **total derivative** associated to the partial derivative vector fields, which are locally expressed as

$$\frac{d}{dx^\mu} = \frac{\partial}{\partial x^\mu} + z^i_\mu \frac{\partial}{\partial y^i} + z^i_{\mu\nu}\frac{\partial}{\partial z^i_\nu} + \ldots$$

2.2 Jet prolongation of vector fields

Definition 2.1. *A 1-form $\theta \in \Lambda^1(Z)$ is said to be a* ***contact 1-form*** *whenever*

$$(j^1\phi)^*\theta = 0$$

for every section ϕ of π.

If (x^μ, y^i, z^i_μ) is a system of local coordinates on Z, then the contact forms are locally spanned by the 1-forms

$$\theta^i = dy^i - z^i_\mu \, dx^\mu$$

We shall denote by $\mathcal{C}$ the algebraic ideal of the contact forms, and by $\mathcal{I}(\mathcal{C})$ the differential ideal generated by the contact forms, in other words, the ideal of the exterior algebra generated by the contact forms and their differentials.

The distribution determined by the annihilation of the contact forms on Z is called the **Cartan distribution** and it plays a fundamental role, since it is the geometrical structure which distinguishes the holonomic sections (sections which are prolongations of sections of π_{XY}) from arbitrary sections of π_{XZ} (see [4, 39, 40, 41, 42, 59] for more details).

Lemma 2.2. *For any vector field X in Z, the following two conditions are equivalent:*

(i) For every Y in the Cartan distribution $\pounds_X Y$ lies in the Cartan distribution; in other words, X preserves the Cartan distribution.

(ii) X preserves $\mathcal{C}$, in other words, for every $\theta \in \mathcal{C}$, $\pounds_X\theta \in \mathcal{C}$.

If any of the preceding two hold, then X preserves $\mathcal{I}(\mathcal{C})$, in other words, for every $\alpha \in \mathcal{I}(\mathcal{C})$, $\pounds_X\alpha \in \mathcal{I}(\mathcal{C})$.

Definition 2.2. *Given a vector field $\xi_Y \in \mathfrak{X}(Y)$, then its* ***1-jet prolongation*** *is defined as the unique vector field $\xi_Y^{(1)} \in \mathfrak{X}(Z)$ projectable onto ξ_Y by π_{YZ}, and which preserves the Cartan distribution (in other words, $\pounds_{\xi_Y^{(1)}}\theta \in \mathcal{C}$ for every contact form θ).*

If ξ_Y is locally expressed as

$$\xi_Y = \xi^\mu_Y \frac{\partial}{\partial x^\mu} + \xi^i_Y \frac{\partial}{\partial y^i}$$

then the 1-jet prolongation of ξ_Y must have the following form

$$\xi_Y^{(1)} = \xi^\mu_Y \frac{\partial}{\partial x^\mu} + \xi^i_Y \frac{\partial}{\partial y^i} + \left(\frac{d\xi^i_Y}{dx^\mu} - z^i_\nu \frac{d\xi^\nu_Y}{dx^\mu} \right) \frac{\partial}{\partial z^i_\mu} \tag{1}$$

Assume that the local expression of $\xi_Y^{(1)}$ is

$$\xi_Y^{(1)} = \xi^\mu_Y \frac{\partial}{\partial x^\mu} + \xi^i_Y \frac{\partial}{\partial y^i} + \xi^i_{\mu Y} \frac{\partial}{\partial z^i_\mu} \tag{2}$$

In order to see that (2) has the form (1), pick $i \in \{1, 2, \ldots, m\}$, and impose the second condition $\pounds_{\xi_Y^{(1)}} \theta^i \in \mathcal{C}$. We have

$$\begin{aligned}\pounds_{\xi_Y^{(1)}} \theta^i &= \frac{\partial \xi_Y^i}{\partial x^\mu} dx^\mu + \frac{\partial \xi_Y^i}{\partial y^j} dy^j - \xi_{\mu Y}^i dx^\mu - z_\mu^i (\frac{\partial \xi_Y^\mu}{\partial x^\nu} dx^\nu + \frac{\partial \xi_Y^\mu}{\partial y^j} dy^j) \\ &= (\frac{\partial \xi_Y^i}{\partial y^j} - z_\mu^i \frac{\partial \xi_Y^\mu}{\partial y^j}) dy^j - (-\frac{\partial \xi_Y^i}{\partial x^\nu} + \xi_{\nu Y}^i + z_\mu^i \frac{\partial \xi_Y^\mu}{\partial x^\nu}) dx^\nu\end{aligned}$$

Therefore

$$-\frac{\partial \xi_Y^i}{\partial x^\nu} + \xi_{\nu Y}^i + z_\mu^i \frac{\partial \xi_Y^\mu}{\partial x^\nu} = z_\nu^j (\frac{\partial \xi_Y^i}{\partial y^j} - z_\mu^i \frac{\partial \xi_Y^\mu}{\partial y^j})$$

and we get

$$\xi_{\mu Y}^i = \frac{d\xi_Y^i}{dx^\mu} - z_\nu^i \frac{d\xi_Y^\nu}{dx^\mu}.$$

Vertical lifting is a Lie algebra homomorphism, as we can see in

Proposition 2.3. *For every $\xi, \zeta \in \mathfrak{X}(Y)$,*

$$[\xi, \zeta]^{(1)} = [\xi^{(1)}, \zeta^{(1)}]$$

Proof. $[\xi^{(1)}, \zeta^{(1)}]$ obviously projects onto $[\xi, \zeta]$, and if α is a contact form, then

$$\pounds_{[\xi^{(1)}, \zeta^{(1)}]} \alpha = \pounds_{\xi^{(1)}} \pounds_{\zeta^{(1)}} \alpha - \pounds_{\zeta^{(1)}} \pounds_{\xi^{(1)}} \alpha$$

which is obviously an element of $\mathcal{C}$. ∎

If ξ_Y is projectable onto a vector field $\xi_X \in \mathfrak{X}(X)$, there is a natural alternative way of defining its 1-jet prolongation, which will be used afterwards. If ξ_Y projects onto ξ_X, having flows Φ_t^Y and Φ_t^X respectively, then $\Phi_t^Z : Z \longrightarrow Z$ defined by $\Phi_t^Z(j_x^1 \phi) = j^1_{\Phi_t^X(x)} (\Phi_t^Y \circ \phi \circ (\Phi_t^X)^{-1})$ is the flow of the 1-jet prolongation of ξ_Y (see [67] for further details).

Lemma 2.4. *For every π_{XY}-projectable vector field $\xi_Y \in \mathfrak{X}(Y)$ and for any form $\alpha \in \bigwedge Z$, and any section $\phi : X \longrightarrow Y$ of π, we have*

$$\left.\frac{d}{dt}\right|_{t=0} (j^1(\Phi_t^Y \circ \phi \circ (\Phi_t^X)^{-1}))^* \alpha = (j^1 \phi)^* \pounds_{\xi_Y^{(1)}} (\alpha)$$

where Φ_t^Y and Φ_t^X are the flows induced by ξ_Y and its projection onto X, respectively.

The proof of this lemma follows in a similar way to the one of Lemma 4.4.5 in [67].

2.3 Lagrangian form. Poincaré-Cartan forms

For first order field theories, the dynamical evolution of a Lagrangian system is described by a **Lagrangian form** $\mathcal{L}$ defined on Z, which is a semibasic $(n+1)$-form in Z respect to the π_{XZ} projector (in other words, it is annihilated when applied to at least one π_{XZ}-vertical vector). This allows us to define the **Lagrangian** function as the unique function L such that $\mathcal{L} = L\eta$.

Let us introduce the following local notation, that we shall often use.

Definition 2.3. *We denote by*

$$\hat{p}_i^\mu := \frac{\partial L}{\partial z_\mu^i}$$

and by

$$\hat{p} := L - z_\mu^i \hat{p}_i^\mu$$

Definition 2.4. *For a given Lagrangian form $\mathcal{L}$ and a volume form η we define the* ***Poincaré-Cartan*** $(n+1)$***-form*** *as*

$$\Theta_L := \mathcal{L} + (S_\eta)^*(dL) \tag{3}$$

In induced coordinates, it has the following expression

$$\begin{aligned}\Theta_L &= \Big(L - z_\mu^i \frac{\partial L}{\partial z_\mu^i}\Big) d^{n+1}x + \frac{\partial L}{\partial z_\mu^i} dy^i \wedge d^n x_\mu \\ &= (\hat{p} dx^\mu + \hat{p}_i^\mu dy^i) \wedge d^n x_\mu \\ &= \mathcal{L} + \hat{p}_i^\mu \theta^i \wedge d^n x_\mu\end{aligned}$$

From this form, we can also define its differential

Definition 2.5. *The* ***Poincaré-Cartan*** $(n+2)$***-form*** *is defined as*

$$\Omega_L := -d\Theta_L .$$

In induced coordinates is expressed as follows

$$\begin{aligned}\Omega_L &= -(dy^i - z_\mu^i dx^\mu) \wedge \Big(\frac{\partial L}{\partial y^i} d^{n+1}x - d\Big(\frac{\partial L}{\partial z_\mu^i}\Big) \wedge d^n x_\mu\Big) \\ &= (d\hat{p} \wedge dx^\mu + d\hat{p}_i^\mu \wedge dy^i) \wedge d^n x_\mu \\ &= -\theta^i \wedge \Big(\frac{\partial L}{\partial y^i} d^{n+1}x - d\hat{p}_i^\mu \wedge d^n x_\mu\Big)\end{aligned}$$

Remark 2.5. *A different choice for the volume form η does not produce changes in the Poincaré-Cartan forms. In fact, if we replace η with a new volume form $\bar{\eta} = F\eta$, where F is a non-vanishing function, we would have $\mathcal{L} = L\eta = \bar{L}\bar{\eta}$, with $\bar{L} = L/F$ and using the preceding computations we finally get $\Theta_L = \Theta_{\bar{L}}$. Thus, we could use the notation $\Theta_{\mathcal{L}}$ and $\Omega_{\mathcal{L}}$ (see [11]).*

At this point, we have to introduce an extra hypothesis on the boundary condition $B \subseteq \partial Z$, that represents boundary conditions on the solutions, which is the existence of an n-form Π on B such that

$$i_B^* \Theta_L = d\Pi$$

where $i_B : B \longrightarrow Z$ is the inclusion map (see [3]).

We can deduce the following properties

Proposition 2.6. *The following holds:*
(i) $(j^1\phi)^* \mathcal{L}_{\xi_Y^{(1)}}(\mathcal{L}) = (j^1\phi)^* \mathcal{L}_{\xi_Y^{(1)}}(\Theta_L)$
(ii) For any $z \in Z$ and every two π_{XZ}-vertical tangent vectors $v, w \in \mathcal{V}_z\pi_{YZ}$,

$$\iota_v \iota_w (\Theta_L)_z = 0$$

(iii) For any $z \in Z$ and every three π_{XZ}-vertical tangent vectors $u, v, w \in \mathcal{V}_z\pi_{YZ}$,

$$\iota_u \iota_v \iota_w (\Omega_L)_z = 0$$

The following proposition will be useful later.

Proposition 2.7. *If σ is a section of π_{XZ} and ξ is a vector field in Z tangent to σ, then*

$$\sigma^*(\iota_\xi \Omega_L) = 0$$

Proof. $\xi = T\sigma(\lambda)$ along σ for certain $\lambda \in \mathfrak{X}(X)$. Thus,

$$\sigma^*(\iota_\xi \Omega_L) = \sigma^*(\iota_{T\sigma(\lambda)} \Omega_L) = \iota_\lambda(\sigma^* \Omega_L) = 0$$

as $\sigma^*\Omega_L = 0$. ∎

2.4 Calculus of variations. Euler-Lagrange equations

The previously introduced geometric objets will take part in the geometric description of the dynamics of field theories, more precisely in the Euler-Lagrange equations, that are traditionally obtained from a variational problem.

The dynamics of the system is given by sections ϕ of π_{XY} which verify the boundary condition $(j^1\phi)(\partial X) \subseteq B$ and that extremise the **action integral**

$$S(\phi) = \int_{(j^1\phi)(C)} \mathcal{L}$$

where C is a compact $(n+1)$-dimensional submanifold of X.

Variations of such sections are introduced by small perturbations of certain section ϕ along the trajectories of a vertical or, in general, a projectable vector field ξ_Y; in other words, if Φ_t^Y is the flow of ξ_Y and Φ_X the flow of its projection, defines the **variations** of ϕ as the sections $\phi_t := \Phi_t^Y \circ \phi \circ \Phi_{-t}^X$.

Definition 2.6. *A section $\phi \in \Gamma(\pi)$ is an* ***extremal*** *of S if*

$$\frac{d}{dt}\Big|_{t=0} \int_{(j^1\phi_t)(C)} \mathcal{L} = \frac{d}{dt}\Big|_{t=0} \int_C (j^1\phi_t)^* \mathcal{L} = 0$$

for any compact $(n+1)$-dimensional submanifold C of X, and for every projectable vector field $\xi_Y \in \mathfrak{X}(Y)$

Lemma 2.4 allows us to rewrite to extremality condition as

$$\int_C (j^1\phi)^* \pounds_{\xi_Y^{(1)}}(\mathcal{L}) = 0 \tag{4}$$

Theorem 2.8. *If ϕ is an extremal of L, then for every $(n+1)$-dimensional compact submanifold C of X, such that $\phi(C)$ lies in a single coordinate domain (x^μ, y^i), and for every projectable vector field ξ_Y on Y we have*

$$\begin{aligned} 0 = &\int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu}\frac{\partial L}{\partial z^i_\mu}\right] (\xi^i_Y - z^i_\nu \xi^\nu_Y)\eta \\ &+ \int_{\partial C} (j^1\phi)^* (\iota_{\xi_Y^{(1)}} \Theta_L) \end{aligned}$$

Whenever ϕ is an extremal for the variational problem with fixed value at the boundary of C, then ϕ satisfies the ***Euler-Lagrange equations***

$$(j^2\phi)^* \left(\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu}\frac{\partial L}{\partial z^i_\mu}\right) = 0, \qquad 1 \le i \le m$$

Proof. A computation on the previous formula gives us

$$\begin{aligned}
\int_C (j^1\phi)^* \pounds_{\xi_Y^{(1)}}(\mathcal{L}) &= \int_C (j^1\phi)^* \xi_Y^{(1)}(L)\eta + \int_C (j^1\phi)^* L(\pounds_{\xi_Y^{(1)}}(\eta)) \\
&= \int_C (j^1\phi)^* \xi_Y^\mu \frac{\partial L}{\partial x^\mu}\eta + \int_C (j^1\phi)^* \xi_Y^i \frac{\partial L}{\partial y^i}\eta \\
&+ \int_C (j^1\phi)^* \left[\frac{d}{dx^\mu}\xi_Y^i - z_\nu^i \frac{d}{dx^\mu}\xi_Y^\nu\right] \frac{\partial L}{\partial z_\mu^i}\eta + \int_C (j^1\phi)^* L(\pounds_{\xi_Y^{(1)}}(\eta)) \\
&= \int_C (j^1\phi)^* \xi_Y^\mu \frac{\partial L}{\partial x^\mu}\eta + \int_C (j^1\phi)^* \xi_Y^i \frac{\partial L}{\partial y^i}\eta \\
&+ \int_C (j^2\phi)^* \frac{d}{dx^\mu}\left[\xi_Y^i - z_\nu^i \xi_Y^\nu\right] \frac{\partial L}{\partial z_\mu^i}\eta + \int_C (j^2\phi)^* \xi_Y^\nu \frac{dz_\nu^i}{dx^\mu}\frac{\partial L}{\partial z_\mu^i}\eta \\
&+ \int_C (j^1\phi)^* L \frac{d\xi_Y^\mu}{dx^\mu}\eta \\
&= \int_C (j^1\phi)^* \xi_Y^\mu \frac{\partial L}{\partial x^\mu}\eta + \int_C (j^1\phi)^* \xi_Y^i \frac{\partial L}{\partial y^i}\eta \\
&+ \int_C (j^2\phi)^* \frac{d}{dx^\mu}\left[\xi_Y^i - z_\nu^i \xi_Y^\nu\right] \frac{\partial L}{\partial z_\mu^i}\eta + \int_C (j^2\phi)^* \xi_Y^\nu \frac{dz_\nu^i}{dx^\mu}\frac{\partial L}{\partial z_\mu^i}\eta \\
&+ \int_{\partial C} (j^1\phi)^* L\xi_Y^\mu d^n x_\mu - \int_C (j^1\phi)^* \xi_Y^\mu \frac{\partial L}{\partial x^\mu}\eta - \int_C (j^1\phi)^* z_\mu^i \frac{\partial L}{\partial y^i}\xi_Y^\mu \eta \\
&- \int_C (j^2\phi)^* \xi_Y^\mu \frac{dz_\nu^i}{dx^\mu}\frac{\partial L}{\partial z_\nu^i}\eta \\
&= \int_C (j^1\phi)^* \frac{\partial L}{\partial y^i}(\xi_Y^i - z_\mu^i \xi_Y^\mu)\eta + \int_C (j^2\phi)^* \frac{d}{dx^\mu}\left[\xi_Y^i - z_\nu^i \xi_Y^\nu\right]\frac{\partial L}{\partial z_\mu^i}\eta \\
&+ \int_{\partial C} (j^1\phi)^* L\xi_Y^\mu d^n x_\mu \\
&= \int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu}\frac{\partial L}{\partial z_\mu^i}\right](\xi_Y^i - z_\nu^i \xi_Y^\nu)\eta \\
&+ \int_{\partial C} (j^1\phi)^* \left[(\xi_Y^i - z_\nu^i \xi_Y^\nu)\frac{\partial L}{\partial z_\mu^i} + L\xi_Y^\mu\right] d^n x_\mu
\end{aligned}$$

The condition of fixed value at the boundary of C means $\xi_Y^\mu|_{\partial C} = \xi_Y^i|_{\partial C} = 0$, therefore we have

$$0 = \int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu}\frac{\partial L}{\partial z_\mu^i}\right](\xi_Y^i - z_\nu^i \xi_Y^\nu)\eta$$

for arbitrary ξ_Y^μ and ξ_Y^i, whence we obtain the Euler-Lagrange equations. ∎

Lemma 2.9. *If ϕ is a section of π_{XY} and ξ is a π_{YZ} vertical vector field in Z, then*

$$(j^1\phi)^*(\iota_\xi \Omega_L) = 0$$

Proof. ξ has components $(0, 0, w_\mu^i)$, and an easy computation shows that

$$\iota_\xi \Omega_L = -w^j_\nu \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} (\theta^i \wedge d^n x_\mu) \in \mathcal{I}(\mathcal{C})$$

which vanishes when pulled back by a 1-jet prolongation of a section of π_{XY}. ∎

Proposition 2.10. ***(Intrinsic version of Euler-Lagrange equations)*** *A section $\phi \in \Gamma(\pi)$ is an extremal of S if and only if*

$$(j^1\phi)^*(\iota_\xi \Omega_L) = 0$$

for every vector field ξ on Z.

Proof. We have that

$$\int_C (j^1\phi)^* \mathcal{L}_{\xi_Y^{(1)}} \mathcal{L} = \int_C (j^1\phi)^* \mathcal{L}_{\xi_Y^{(1)}} \Theta_L = -\int_C (j^1\phi)^* \iota_{\xi_Y^{(1)}} \Omega_L + \int_{\partial C} (j^1\phi)^* \iota_{\xi_Y^{(1)}} \Theta_L$$

Therefore,

$$-\int_C (j^1\phi)^* \iota_{\xi_Y^{(1)}} \Omega_L = \int_C (j^2\phi)^* \left[\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu} \frac{\partial L}{\partial z^i_\mu} \right] (\xi^i_Y - z^i_\nu \xi^\nu_Y)\eta$$

for every projectable vector field ξ_Y on Y. Then, Euler-Lagrange equations are satisfied in every C if and only if

$$(j^1\phi)^* \iota_{\xi_Y^{(1)}} \Omega_L = 0$$

for every projectable vector field ξ_Y on Y, in every compact C of X. Now different local solutions can be glued together using partitions of unity, so that we get that

$$(j^1\phi)^* \iota_{\xi_Y^{(1)}} \Omega_L = 0$$

is the expression for global sections ϕ.

Finally, any general vector field ξ_Z may be decomposed into a vector field tangent to $j^1\phi$, the lift of a π_{XY}-vertical vector field on Y and a π_{YZ}-vertical vector field. Using the preceding lemma, and Proposition 2.7, we get the result. ∎

2.5 Regular Lagrangians. De Donder equations

In some cases, we shall need to assume extra regularity conditions on the Lagrangian function:

Definition 2.7. *For a Lagrangian function $L : Z \longrightarrow \mathbb{R}$, it is defined its* ***Hessian matrix***

$$\left(\frac{\partial^2 L}{\partial z^\alpha_i \partial z^\beta_j} \right)_{\alpha,\beta,i,j}$$

The Lagrangian is said to be ***regular at*** *a point whenever such matrix is regular at that point, and* ***regular*** *whenever it is regular everywhere.*

When the Lagrangian is regular, the implicit function theorem allows us to introduce new coordinates for Z, called **Darboux coordinates** [52, 57, 58], namely $(x^\mu, y^i, \hat{p}_i^\mu)$, which will also be very convenient to relate the Lagrangian formalism to Hamiltonian formalism.

We introduce the De Donder equations, closely related to the Euler-Lagrange equations.

Definition 2.8. *The following equation on sections σ of π_{XZ} is called the **De Donder equations**:*

$$\sigma^*(\iota_\xi \Omega_L) = 0 \qquad \forall \xi \in \mathfrak{X}(Z) \tag{5}$$

Sections satisfying the De Donder equations and in addition the boundary condition $\sigma(\partial X) \subseteq B$ are called solutions of the De Donder equations.

From proposition (2.7), we deduce that De Donder equations can be equivalently restated in terms of π_{XZ}-vertical vector fields. In local coordinates, if $\sigma(x^\mu) = (x^\mu, \sigma^i(x^\mu), \sigma^i_\nu(x^\mu))$ for any $\xi = v^i \frac{\partial}{\partial y^i} + w^i_\mu \frac{\partial}{\partial z^i_\mu}$ the equation is written as

$$\begin{aligned} 0 = &- v^i \left(\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \frac{\partial \sigma^j}{\partial x^\mu} \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - \frac{\partial \sigma^j_\mu}{\partial x^\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + \left(\frac{\partial \sigma^j}{\partial x^\mu} - \sigma^j_\mu \right) \frac{\partial^2 L}{\partial y^i \partial z^j_\mu} \right) \\ &+ w^i_\mu \left(\left(\frac{\partial \sigma^j}{\partial x^\nu} - \sigma^j_\nu \right) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right), \end{aligned}$$

or, in other words,

$$\left. \begin{aligned} \frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \frac{\partial \sigma^j}{\partial x^\mu} \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - \frac{\partial \sigma^j_\mu}{\partial x^\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + \left(\frac{\partial \sigma^j}{\partial x^\mu} - \sigma^j_\mu \right) \frac{\partial^2 L}{\partial y^i \partial z^j_\mu} &= 0 \\ \left(\frac{\partial \sigma^j}{\partial x^\nu} - \sigma^j_\nu \right) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} &= 0 \end{aligned} \right\}$$

From the expression above, we immediately deduce that

Proposition 2.11. *If the Lagrangian is regular, then if a section $\sigma : X \longmapsto Z$ of π_{XZ} is a solution of the De Donder equations, then there is a section $\phi : X \longrightarrow Y$ of π_{XY} such that $\sigma = j^1\phi$. Furthermore, ϕ is a solution of the Euler-Lagrange equations.*

Therefore, for regular Lagrangians, the solutions of the De Donder equations provide the information about the dynamics of the system.

2.6 The De Donder equations in terms of Ehresmann connections

Suppose that we have a connection Γ in $\pi : Z \longrightarrow X$, with horizontal projector $\mathbf{h}$. Here, Γ is a connection in the sense of Ehresmann, that is, Γ defines a horizontal complement of the

vertical bundle $\mathcal{V}\pi_{XZ}$. The horizontal projector has the following local expression:

$$\begin{cases} \mathbf{h}(\dfrac{\partial}{\partial x^\mu}) & = \dfrac{\partial}{\partial x^\mu} + \Gamma^i_\mu \dfrac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu} \dfrac{\partial}{\partial z^i_\nu} \\ \mathbf{h}(\dfrac{\partial}{\partial y^i}) & = 0 \\ \mathbf{h}(\dfrac{\partial}{\partial z^i_\mu}) & = 0 \end{cases}$$

A direct computation shows that

$$\begin{aligned} \iota_{\mathbf{h}}\Omega_L = n\Omega_L - \sum_i \Bigg[& \frac{\partial L}{\partial y^i} - \sum_\nu \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \sum_{\nu,j} \Gamma^j_\nu \frac{\partial^2 L}{\partial y^j \partial z^i_\nu} \\ & - \sum_{\nu,\mu,j} \Gamma^j_{\mu\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + \sum_{\nu,j} (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial y^i \partial z^j_\nu} \Bigg] dy^i \wedge d^{n+1}x \\ & - \sum_{\mu,i} \left(\sum_{\nu,j} (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right) dz^i_\mu \wedge d^{n+1}x \end{aligned}$$

from where we can state the following.

Proposition 2.12. *Let Γ be a connection with horizontal projector $\boldsymbol{h}$ verifying*

$$\iota_{\mathbf{h}}\Omega_L = n\Omega_L \tag{6}$$

If σ is a horizontal local integral section of Γ, then σ is a solution of the De Donder equations.

Proof. $\mathbf{h}$ satisfies (6) if and only if

$$\left. \begin{aligned} \frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\nu \partial z^i_\nu} - \Gamma^j_\nu \frac{\partial^2 L}{\partial y^j \partial z^i_\nu} - \Gamma^j_{\mu\nu} \frac{\partial^2 L}{\partial z^j_\mu \partial z^i_\nu} + (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial y^i \partial z^j_\nu} &= 0 \\ (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} &= 0 \end{aligned} \right\}$$

If $\sigma(x^\mu) = (x^\mu, \sigma^i(x^\mu), \sigma^i_\nu(x^\mu))$ is a horizontal local integral section of Γ, then we have that

$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = T\sigma(\frac{\partial}{\partial x^\mu}) \tag{7}$$

which means that $\Gamma^i_\mu = \dfrac{\partial \sigma^i}{\partial x^\mu}$ and $\Gamma^i_{\mu\nu} = \dfrac{\partial \sigma^i_\nu}{\partial x^\mu}$, and therefore (6) becomes the De Donder equations in coordinates.

Local solutions can be glued together using partitions of unity. ∎

If we consider boundary conditions, then the connection $\mathbf{h}$ induces a connection $\partial\mathbf{h}$ in the fibration $\pi_{\partial XB} : B \longrightarrow \partial X$, since we are considering sections $\sigma \in \Gamma(\pi_{XZ})$ such that $\sigma(\partial X) \subseteq B$.

In this way, the equation (6) becomes $\iota_{\mathbf{h}}\Omega_L = n\Omega_L$ with the additional condition that $\mathbf{h}$ induces $\partial\mathbf{h}$ (or equivalently $\mathbf{h}_z(T_zB) \subseteq T_zB$ for all $z \in B$).

In the regular case (or for semiholonomic connections, that is $\Gamma^i_\mu = z^i_\mu$), two of these solutions differ by a $(1,1)-$tensor field T, locally given by

$$T = T^i_{\mu\nu} dx^\nu \otimes \frac{\partial}{\partial z^i_\mu}$$

and verifying

$$T^i_{\mu\nu} \frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} = 0$$

Remark 2.13. *An alternative approach may be considered if we express (6) for horizontal integrable distributions in terms of multivector fields generating those distributions. For further details, see [12, 13, 15, 16, 17, 18] and [19, 61, 62].*

2.7 The singular case

For a singular Lagrangian L, one cannot expect to find globally defined solutions; in general, if such connection $\mathbf{h}$ exists, it does so only along a submanifold Z_f of Z.

In [48, 49] the authors have developed a constraint algorithm which extends the Dirac-Bergmann-Gotay-Nester-Hinds algorithm for Mechanics (see [26, 30, 31], and also [43, 46] for more recent developments).

Put $Z_1 = Z$. We then consider the subset

$$\begin{aligned} Z_2 &= \{z \in Z \mid \exists \mathbf{h}_z : T_zZ \longrightarrow T_zZ \quad \text{linear such that } \mathbf{h}_z^2 = \mathbf{h}_z, \ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z, \\ &\quad i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z), \text{and for } z \in B, \text{we also have } \mathbf{h}_z(T_zB) \subseteq T_zB\}. \end{aligned}$$

If Z_2 is a submanifold, then there are solutions but we have to include the tangency condition, and consider a new step (denoting $B_2 = B \cap Z_2$, and in general, $B_r = B \cap Z_r$):

$$\begin{aligned} Z_3 &= \{z \in Z_2 \mid \exists \mathbf{h}_z : T_zZ \longrightarrow T_zZ_2 \quad \text{linear such that } \mathbf{h}_z^2 = \mathbf{h}_z, \ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z, \\ &\quad i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z), \text{and for } z \in B_2, \text{we also have } \mathbf{h}_z(T_zB_2) \subseteq T_zB_2\}. \end{aligned}$$

If Z_3 is a submanifold of Z_2, but $\mathbf{h}_z(T_zZ)$ is not contained in T_zZ_3 and $\mathbf{h}_z(T_zB)$ is not contained in T_zB for $z \in B$, we go to the third step, and so on. In the favourable case, we would obtain a final constraint submanifold Z_f of non-zero dimension, and a connection for the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f (in fact, a family of connections)

with horizontal projector **h** which is a solution of equation (6), and, in addition, it satisfies the boundary condition .

There is an additional problem, since our connection would be a solution of the De Donder problem, but not a solution of the Euler-Lagrange equations. This problem is solved constructing a submanifold of Z_f where such a solution exists (see [48, 49] for more details).

2.8 Multisymplectic forms. Brackets

Definition 2.9. *[27] A* ***multisymplectic form*** Ω *in a manifold* M *is a closed* k*-form* $(k > 1)$ *having the following non-degeneracy property:*

$$\iota_v \Omega = 0 \text{ if and only if } v = 0 \qquad \forall v \in T_x M, \forall x \in M$$

A ***multisymplectic manifold*** *is a manifold endowed with a multisymplectic form.*

The properties of multisymplectic manifolds have been widely explored in [5, 52, 57, 58].

Proposition 2.14. *For* $n > 0$*, the Lagrangian* L *is regular if and only if* Ω_L *is a multisymplectic form*

Proof. As the Lagrangian is regular, we can use Darboux coordinates $(x^\mu, y^i, \hat{p}_i^\mu)$ (see also Definition 2.3), and the expression of Ω_L in these coordinates was stated shortly after its definition. From the following computations:

$$\begin{aligned}
\iota_{\partial/\partial x^\nu}\Omega_L &= -\frac{\partial \hat{p}}{\partial x^\nu} d^{n+1}x + d\hat{p} \wedge d^n x_\nu + d\hat{p}_i^\mu \wedge dy^i \wedge d^{n-1}x_{\mu\nu} \\
&= \frac{\partial \hat{p}}{\partial y^i} dy^i \wedge d^n x_\nu + \frac{\partial \hat{p}}{\partial \hat{p}_i^\mu} d\hat{p}_i^\mu \wedge d^n x_\nu + d\hat{p}_i^\mu \wedge dy^i \wedge d^{n-1}x_{\mu\nu} \\
\iota_{\partial/\partial y^j}\Omega_L &= \frac{\partial \hat{p}}{\partial y^j} d^{n+1}x - d\hat{p}_j^\mu \wedge d^n x_\mu \\
\iota_{\partial/\partial \hat{p}_j^\nu}\Omega_L &= \frac{\partial \hat{p}}{\partial \hat{p}_j^\nu} d^{n+1}x + dy^j \wedge d^n x_\nu
\end{aligned}$$

if we have $\xi = A^\nu \frac{\partial}{\partial x^\nu} + B^j \frac{\partial}{\partial y^j} + C_j^\nu \frac{\partial}{\partial \hat{p}_j^\nu}$ then

$$\begin{aligned}
\iota_\xi \Omega_L = &\left(B^j \frac{\partial \hat{p}}{\partial y^j} - C_j^\nu \frac{\partial \hat{p}}{\partial \hat{p}_j^\nu} \right) d^{n+1}x + \left(A^\nu \frac{\partial \hat{p}}{\partial \hat{p}_j^\mu} - \delta_\mu^\nu B^j \right) d\hat{p}_j^\mu \wedge d^n x_\nu \\
&+ \left(A^\nu \frac{\partial \hat{p}}{\partial y^j} - C_j^\nu \right) dy^j \wedge d^n x_\nu + A^\nu d\hat{p}_i^\mu \wedge dy^i \wedge d^{n-1}x_{\mu\nu}
\end{aligned}$$

Therefore, if $\iota_\xi \Omega_L = 0$ and $n > 0$, then from the last term of the expression above, $A^\nu = 0$, and we easily get that the rest of terms B^j and C_j^ν vanish as well. The converse is proven in a similar manner. ∎

Remark 2.15. *The case $n = 0$ has many differences from the case $n > 0$, and corresponds to the case of the time-dependent Lagrangian mechanics (see [55]). In this case, the regularity of L implies that (Z, Ω_L, dt) (where $dt = \eta$ is the volume form) is a cosymplectic manifold. The connection equation reduces to*

$$\iota_{\boldsymbol{h}}\Omega_L = 0$$

where if we call $\tau = \frac{\partial}{\partial t}$ (so that $\langle \eta | \tau \rangle = 1$), then the horizontal projector $\boldsymbol{h}$ can be written in coordinates as follows

$$\mathbf{h}(\tau) = \tau + h^i \frac{\partial}{\partial q^i} + h'^i \frac{\partial}{\partial v^i}$$

(for $q^i = y^i, v^i = z_0^i$). Sections of π_{XY} are curves on Y, and Z can be embedded in TY.

One obtains from De Donder equations that $h'^i = \frac{\partial h^i}{\partial t}$, and that $h(\tau)$ verifies the time dependent Euler-Lagrange equations on $J^1\pi$. Furthermore, for a $(1,1)$-tensor field h on $J^1\pi$, being the horizontal projector of a distribution solution of

$$\iota_{\boldsymbol{h}}\Omega_L = 0$$

is equivalent to having $\xi = \mathbf{h}(\tau)$ which verifies

$$\begin{aligned} \iota_\xi \Omega_L &= 0 \\ \iota_\xi \eta &= 1 \end{aligned}$$

From now on within this section, we shall suppose that $n > 0$.

With multisymplectic structures we can define Hamiltonian vector fields and forms as we did for symplectic structures. However, existence is no longer guaranteed.

Definition 2.10. *Let α be a n-form in Z. A vector field X_α is called a* ***Hamiltonian vector field*** *for α, and we say that α is* ***Hamiltonian*** *whenever*

$$d\alpha = \iota_{X_\alpha}\Omega_L$$

If L is regular, then the non-degeneracy of Ω_L guarantees that a Hamiltonian vector field, if it exists, is unique. Otherwise, we cannot guarantee its existence, and the Hamiltonian vector field is defined up to an element in the kernel of Ω_L.
Also note that two forms that differ by a closed form have the same Hamiltonian vector fields.

Definition 2.11. *If α and β are two Hamiltonian n-forms for which there exist the corresponding Hamiltonian vector fields X_α, X_β, then we can define the* ***bracket operation*** *as follows:*

$$\{\alpha, \beta\} = \iota_{X_\beta}\iota_{X_\alpha}\Omega_L$$

We also have the following result:

Proposition 2.16. *If and α, β are Hamiltonian n-forms which have a Hamiltonian vector fields X_α and X_β respectively, then $\{\alpha, \beta\}$ is a Hamiltonian n-form which has associated Hamiltonian vector field $[X_\alpha, X_\beta]$. In other words,*

$$X_{\{\alpha,\beta\}} = [X_\alpha, X_\beta]$$

Proof.

$$\begin{aligned}
\iota_{[X_\alpha,X_\beta]}\Omega_L &= \mathcal{L}_{X_\alpha}\iota_{X_\beta}\Omega_L - \iota_{X_\beta}\mathcal{L}_{X_\alpha}\Omega_L \\
&= \mathcal{L}_{X_\alpha}d\beta - \iota_{X_\beta}d\iota_{X_\alpha}\Omega_L - \iota_{X_\beta}\iota_{X_\alpha}d\Omega_L \\
&= d\iota_{X_\alpha}d\beta - \iota_{X_\beta}dd\alpha \\
&= -d\iota_{X_\alpha}\iota_{X_\beta}\Omega_L \\
&= d\{\alpha, \beta\},
\end{aligned}$$

and, by uniqueness, we obtain the desired result. ∎

The properties of this brackets have been widely studied in [6, 19, 25].

3 Hamiltonian formalism

3.1 Dual jet bundle

At the beginning of our discussion, we briefly listed the different approaches to the notion of jet bundle, where one of these is to consider it certain structure of affine bundle over Y.

The dual affine bundle of the jet bundle is called **dual jet bundle**, and it is usually denoted by $(J^1\pi)^*$, that we shall denote by Z^*. An alternative construction of such bundle is given here.

Definition 3.1. *Consider the family of spaces of forms*

$$\Lambda^{n+1}_r Y := \{\sigma \in \Lambda^{n+1} Y \mid \iota_{V_1} \dots \iota_{V_r}\sigma = 0, \forall V_i \; \pi - vertical \; 1 \leq i \leq r\}$$

*In particular, the elements of $\Lambda^{n+1}_1 Y$ are called **semibasic** $(n+1)$**-forms**. It is a fiber bundle over Y of rank $(n+1+m+1)$, and which elements can be locally expressed as $p(x,y)d^{n+1}x$.*

Similarly, $\Lambda^{n+1}_2 Y$ is a vector bundle over Y of rank $(n+1+m+(n+1)m+1)$, having $\Lambda^{n+1}_1 Y$ as subbundle, and which elements can be locally expressed as $p(x,y)d^{n+1}x + p^\mu_i(x,y)dy^i \wedge d^n x_\mu$. The natural projection will be called:

$$\nu_r : \Lambda^{n+1}_r Y \longrightarrow Y$$

The quotient bundle

$$Z^* = (J^1\pi)^* := \Lambda_2^{n+1}Y/\Lambda_1^{n+1}Y$$

is a vector bundle over Y *of rank* $n+1+m+(n+1)m$ *which elements can be locally expressed as* $p_i^\mu(x,y)dy^i \wedge d^n x_\mu$, *and that is called the* ***dual first order jet bundle****. The canonical projection will be denoted by* $\mu : \Lambda_2^{n+1}Y \longrightarrow Z^*$.

We can define a projection $\pi_{XZ^*} : Z^* \longrightarrow X$, *which is induced by* ν_2 *into the quotient space* Z^*, *composed with* π_{XY}.

Definition 3.2. *The manifold* $\Lambda_2^{n+1}Y$ *is equipped with the following* $(n+1)$*-form*

$$\Theta_\omega(X_0, \ldots, X_n) := \omega(T\nu_2(X_0), \ldots, T\nu_2(X_n))$$

which is called the ***multimomentum Liouville form****, and has local expression*

$$\Theta = pd^{n+1}x + p_i^\mu dy^i \wedge d^n x_\mu$$

We also define the ***canonical multisymplectic*** $(n+2)$***-form*** *on* $\Lambda_2^{n+1}Y$ *by*

$$\Omega := -d\Theta$$

Notice that Ω is in fact multisymplectic, by a similar argument to that given in Proposition 2.14.

3.2 Lift of vector fields to the dual jet bundle

A vector field ξ_Y on Y, having flow ϕ_t, admits a natural lift to $\Lambda^k Y$ for any k, having flow $(\phi_t^{-1})^*$.

If the vector field ξ_Y is projectable, then the flow preserves $\Lambda_2^{n+1}Y$ and $\Lambda_1^{n+1}Y$, and therefore we can define on $\Lambda_2^{n+1}Y$ a vector field which projects onto a vector field on Z^*, which we shall denote by $\xi_Y^{(1*)}$.

In general, if α is the pull-back to $\Lambda_2^{n+1}Y$ of certain semibasic n-form on Y, locally expressed by

$$\alpha = \alpha^\nu(x^\mu, y^i)d^n x_\nu,$$

the additional condition $\pounds_{\xi_Y^\alpha}\Theta = d\alpha$ imposed to vector fields on $\Lambda^{n+1}Y$ which project to ξ_Y, determines a vector field on $\Lambda^{n+1}Y$ that can be defined on $\Lambda_2^{n+1}Y$.

In other words, we have the following definition.

Definition 3.3. *If* α *is the pull-back to* $\Lambda_2^{n+1}Y$ *of a* π_{XY}*-semibasic form, then the* α*-lift of a vector field* ξ_Y *on* Y *to* $\Lambda_2^{n+1}Y$ *is defined as the unique vector field* ξ_Y^α *satisfying:*

(1) ξ_Y^α *projects onto* ξ_Y

(2) $\pounds_{\xi_Y^\alpha}\Theta = d\alpha$

An easy computation shows that the components $dp(\xi_Y^\alpha) = \xi_Y^p$ and $dp_i^\mu(\xi_Y^\alpha) = \xi_Y^{p_i^\mu}$ are determined by the equations (see also [28, 61]):

$$
\begin{aligned}
\xi_Y^p &= -p\frac{\partial \xi_Y^\mu}{\partial x^\mu} - p_i^\mu \frac{\partial \xi_Y^i}{\partial x^\mu} - \frac{\partial \alpha^\mu}{\partial x^\mu} \\
\xi_Y^{p_i^\mu} &= p_i^\nu \frac{\partial \xi_Y^\mu}{\partial x^\nu} - p_j^\mu \frac{\partial \xi_Y^j}{\partial y^i} - p_i^\mu \frac{\partial \xi_Y^\nu}{\partial x^\nu} - \frac{\partial \alpha^\mu}{\partial y^i}
\end{aligned}
$$

When ξ_Y is π_{XY}-projectable, with flow ϕ_t, then the flow of the 0-lift is precisely $(\phi_t^{-1})^*$.

3.3 Hamilton equations

Definition 3.4. *A **Hamiltonian** form is a section $h : Z^* \longrightarrow \Lambda_2^{n+1}Y$ of the natural projection $\mu : \Lambda_2^{n+1}Y \longrightarrow Z^*$.*

In local coordinates, h is given by

$$h(x^\mu, y^i, p_i^\mu) = (x^\mu, y^i, p = -H(x^\mu, y^i, p_i^\mu), p_i^\mu)$$

*where H is called a **Hamiltonian function**.*

Definition 3.5. *Given a Hamiltonian, we define the following forms in Z^**

$$\Theta_h := h^*\Theta$$

having local expression

$$
\begin{aligned}
\Theta_h &= -H d^{n+1}x + p_i^\mu dy^i \wedge d^n x_\mu \\
&= (-H dx^\mu + p_i^\mu dy^i) \wedge d^n x_\mu
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_h :&= h^*\Omega = -d\Theta_h \\
&= (-dH \wedge +dx^\mu + dp_i^\mu \wedge dy^i) \wedge d^n x_\mu
\end{aligned}
$$

Definition 3.6. *For a given Hamiltonian h, a section $\sigma : X \longrightarrow Z^*$ of π_{XZ^*} is said to satisfy the **Hamilton equations** if*

$$\sigma^*(\iota_\xi \Omega_h) = 0$$

for all vector field ξ on Z^*.

If σ has local expression $\sigma(x^\mu) = (x^\mu, \sigma^i(x^\mu), \sigma_i^\nu(x^\mu))$, then the Hamilton equations are written in coordinates as follows

$$
\begin{aligned}
\frac{\partial \sigma^i}{\partial x^\mu} &= \frac{\partial H}{\partial p_i^\mu} \\
\sum_{\mu=1}^{m} \frac{\partial \sigma_i^\mu}{\partial x^\mu} &= -\frac{\partial H}{\partial y^i}
\end{aligned}
$$

As for the Lagrangian case, we can also consider the case of having a boundary condition given by a subbundle $B^* \subseteq \partial Z^*$ of $\tilde{\pi}_{\partial X \partial Z}$, which imposes a restriction on the possible solutions for the Hamilton equations. The additional requirement for the solutions is naturally that they must satisfy $\sigma(\partial X) \subseteq B^*$, and we also need to assume that

$$i_{B^*}^* \Theta_h = d\Pi^*$$

for certain n-form Π^* on B^*, where $i_{B^*} : B^* \longrightarrow \partial Z^*$ denotes the canonical inclusion.

There is also another formulation of the Hamilton equations in terms of connections.

Suppose that we have a connection Γ (in the sense of Ehresmann) in $\pi_{XZ^*} : Z^* \longrightarrow X$, with horizontal projector $\mathbf{h}$, and having a local expression as follows

$$\begin{cases} \mathbf{h}(\dfrac{\partial}{\partial x^\mu}) & = \dfrac{\partial}{\partial x^\mu} + \Gamma^i_\mu \dfrac{\partial}{\partial y^i} + \Gamma^\nu_{i\mu} \dfrac{\partial}{\partial p^\nu_i} \\ \mathbf{h}(\dfrac{\partial}{\partial y^i}) & = 0 \\ \mathbf{h}(\dfrac{\partial}{\partial p^\mu_i}) & = 0 \end{cases}$$

A direct computation shows that

$$\begin{aligned} \iota_{\mathbf{h}} \Omega_h = n\Omega_h &- \left(\frac{\partial H}{\partial y^i} + \sum_{\mu=1}^{m} \Gamma^\mu_{i\mu} \right) dy^i \wedge d^{n+1}x \\ &+ \left(\frac{\partial H}{\partial p^\mu_i} - \Gamma^i_\mu \right) dp^\mu_i \wedge d^{n+1}x \end{aligned}$$

From where we can state the following.

Proposition 3.1. *Let Γ be a connection with horizontal projector $\boldsymbol{h}$ verifying*

$$\iota_{\mathbf{h}} \Omega_h = n\Omega_h \tag{8}$$

and also the boundary compatibility condition $\mathbf{h}_\alpha(T_\alpha B^*) \subseteq T_\alpha B^*$ for $\alpha \in Z^*$ (i.e., $\boldsymbol{h}$ induces a connection $\partial \boldsymbol{h}$ in the fibration $\pi_{\partial X B^*} : B^* \longrightarrow \partial X$).

If σ is a horizontal integral local section of Γ, then σ is a solution of the Hamilton equations.

Therefore, one can think of the preceding equation as an alternative approach to the Hamilton equations.

3.4 The Legendre transformation

We shall generalize to field theories the notion of Legendre transformation in Classical Mechanics.

Definition 3.7. *Associated to the Lagrangian function we can define the* ***Legendre transformation*** $Leg_L : Z \longrightarrow \Lambda_2^{n+1}Y$ *as follows, given* $\xi_1, \ldots, \xi_n \in (T_{\pi_{YZ}z})Y$,

$$(Leg_L(z))(\xi_1, \ldots, \xi_n) = (\Theta_L)_z(\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$$

where $\tilde{\xi}_i$ *is a tangent vector at* $z \in Z$ *which projects onto* ξ_i.

It is well defined, as $\iota_\xi \Theta_L = 0$ *for* π_{YZ}*-vertical vector fields (see lemma 2.6), and* $\iota_\xi \iota_\zeta Leg_L(z) = 0$ *for* $\xi, \zeta \in \mathcal{V}\pi$, *therefore,* $Leg_L(z) \in \Lambda_2^{n+1}Y$.

In local coordinates,

$$Leg_L(x^\mu, y^i, z^i_\mu) = \left(x^\mu, y^i, p = L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}, p_i^\mu = \frac{\partial L}{\partial z^i_\mu}\right)$$

which shows that Leg_L *is a fibered map over* Y.

For an expression of the Legendre transformation in terms of affine duals, see [28].

Definition 3.8. *We also define the* ***Legendre map*** $leg_L := \mu \circ Leg_L : Z \longrightarrow Z^*$, *which in coordinates has the form:*

$$leg_L(x^\mu, y^i, z^i_\mu) = \left(x^\mu, y^i, p_i^\mu = \frac{\partial L}{\partial z^i_\mu} = \hat{p}_i^\mu\right)$$

From the local expressions of Θ_L, the following proposition is obvious.

Proposition 3.2. *All these facts hold:*

(i) The Lagrangian is regular if and only if then the Legendre map leg_L *is a local diffeomorphism.*

(ii) If we choose a Hamiltonian h, *then we have the following relations:*

$$(Leg_L)^*\Theta = \Theta_L, \qquad (Leg_L)^*\Omega = \Omega_L$$

$$(leg_L)^*\Theta_h = \Theta_L, \qquad (leg_L)^*\Omega_h = \Omega_L$$

Definition 3.9. *A Lagrangian* L *is called* ***hyperregular*** *whenever* leg_L *is a diffeomorphism (and therefore, it is regular). Also assume that* $leg_L^*(\Pi^*) = \Pi$.

We also have the following equivalence theorem, which is a straightforward computation.

Theorem 3.3. *(**equivalence theorem**). Suppose that the Lagrangian is regular. Then if a section σ_1 of π_{XZ} satisfies the De Donder equations*

$$\sigma_1^*(\iota_\xi \Omega_L) = 0 \qquad\qquad \forall \xi \in \mathfrak{X}(Z)$$

then $\sigma_2^* := leg \circ \sigma_1$ verifies the Hamilton equations

$$\sigma_2^*(\iota_\xi \Omega_h) = 0 \qquad\qquad \forall \xi \in \mathfrak{X}(Z^*)$$

Reciprocally, if σ_2 verifies Hamilton equations, then (the locally defined) $\sigma_1 := leg_L^{-1} \circ \sigma_2$ verifies the De Donder equations. Therefore, De Donder equations are equivalent to Hamilton equations.

Remark 3.4. *A rutinary computation also shows that, for a regular Lagrangian, if Γ is a connection solution of* (6) *then $Tleg_L(\Gamma)$ is a solution for the equation in terms of connections on the Hamiltonian side.*

Furthermore, a boundary condition B on Z automatically induces a boundary condition B^* in Z^*, by $leg_L(B) = B^*$, which implies that $Tleg_L(T_z B) \subseteq T_{leg_L(z)} B^*$, and in turn proves that compatible connection projectors relate to each other via the Legendre map.

3.5 Almost regular Lagrangians

When the Lagrangian is not regular then to develop a Hamiltonian counterpart, we need some weak regularity condition on the Lagrangian L, the almost-regularity assumption.

Definition 3.10. *A Lagrangian $L : Z \longrightarrow \mathbb{R}$ is said to be **almost regular** if $Leg_L(Z) = \tilde{M}_1$ is a submanifold of $\Lambda_2^{n+1} Y$, and $Leg_L : Z \longrightarrow \tilde{M}_1$ is a submersion with connected fibers.*

If L is almost regular, we deduce that:

- $M_1 = leg_L(Z)$ is a submanifold of Z^*, and in addition, a fibration over X and Y.
- The restriction $\mu_1 : \tilde{M}_1 \longrightarrow M_1$ of μ is a diffeomorphism.
- The mapping $leg_L : Z \longrightarrow M_1$ is a submersion with connected fibers.

On the hypothesis of almost regularity, we can define a mapping $h_1 = (\mu_1)^{-1} : M_1 \longrightarrow \tilde{M}_1$, and a $(n+2)$-form Ω_{M_1} on M_1 by $\Omega_{M_1} = h_1^*(j^*\Omega)$ considering the inclusion map $j : \tilde{M}_1 \hookrightarrow \Lambda_2^{n+1} Y$. Obviously, we have $leg_1^* \Omega_{M_1} = \Omega_L$, where $j \circ leg_1 = leg_L$ (see Figure 2).

$$
\begin{tikzcd}
 & \tilde{M}_1 = Leg_L(Z) \arrow[r, "\tilde{j}"] \arrow[dd, "\mu_1"] & \Lambda_2^{n+1}Y \arrow[dd, "\mu"] \\
Z \arrow[ur, "Leg_1"] \arrow[dr, "leg_1"'] \arrow[urr, bend left, "Leg_L"] \arrow[drr, bend right, "leg_L"'] & & \\
 & M_1 = leg_L(Z) \arrow[r, "j"] & Z^*
\end{tikzcd}
$$

Figure 2

The Hamiltonian description is now based in the equation

$$i_{\tilde{\mathbf{h}}}\Omega_{M_1} = n\Omega_{M_1} \tag{9}$$

where $\tilde{\mathbf{h}}$ is a connection in the fibration $\pi_{XM_1} : M_1 \longrightarrow X$, and the additional boundary condition for $\tilde{\mathbf{h}}$.

Proceeding as before, we construct a constraint algorithm as follows. First, we denote by $B_1^* = B^* \cap M_1$, and will assume it to be a submanifold of B^* (and in general we shall denote $B_r^* = B^* \cap M_r$, which will also be assumed to be a submanifold of B_{r-1}^*), and we define

$$
\begin{aligned}
M_2 \;=\; & \{\tilde{z} \in M_1 \mid \exists \tilde{\mathbf{h}}_{\tilde{z}} : T_{\tilde{z}}M_1 \longrightarrow T_{\tilde{z}}M_1 \quad \text{linear such that } \; \tilde{\mathbf{h}}_{\tilde{z}}^2 = \tilde{\mathbf{h}}_{\tilde{z}}, \ker \tilde{\mathbf{h}}_{\tilde{z}} = (\mathcal{V}\pi_{XM_1})_{\tilde{z}}, \\
& i_{\tilde{\mathbf{h}}_{\tilde{z}}}\Omega_{M_1}(\tilde{z}) = n\Omega_{M_1}(\tilde{z}), \text{and for } \tilde{z} \in B_1^* \text{we also have } \tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}B_1^*) \subseteq T_{\tilde{z}}B_1^*\}.
\end{aligned}
$$

If M_2 is a submanifold (possibly with boundary) then there are solutions but we have to include the tangency conditions, and consider a new step:

$$
\begin{aligned}
M_3 \;=\; & \{\tilde{z} \in M_2 \mid \exists \tilde{\mathbf{h}}_{\tilde{z}} : T_{\tilde{z}}M_1 \longrightarrow T_{\tilde{z}}M_2 \quad \text{linear such that } \; \tilde{\mathbf{h}}_{\tilde{z}}^2 = \tilde{\mathbf{h}}_{\tilde{z}}, \ker \tilde{\mathbf{h}}_{\tilde{z}} = (\mathcal{V}\pi_{XM_1})_{\tilde{z}}, \\
& i_{\tilde{\mathbf{h}}_{\tilde{z}}}\Omega_{M_1}(\tilde{z}) = n\Omega_{M_1}(\tilde{z}), \text{and for } \tilde{z} \in B^* \cap M_2 \text{we also have } \tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}B^*) \subseteq T_{\tilde{z}}B^*\}.
\end{aligned}
$$

If M_3 is a submanifold of M_2, but $\tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}M_1)$ is not contained in $T_{\tilde{z}}M_3$, and $\tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}B^*)$ is not contained in $T_{\tilde{z}}B^*$ for $z \in B^*$, we go to the third step, and so on. Thus, we proceed further to obtain a sequence of embedded submanifolds

$$... \hookrightarrow M_3 \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow Z^*$$

with boundaries

$$... \hookrightarrow B_3^* \hookrightarrow B_2^* \hookrightarrow B_1^* \hookrightarrow B^*$$

If this constraint algorithm stabilizes, we shall obtain a final constraint submanifold M_f of non-zero dimension and a connection in the fibration $\pi_{XM_1} : M_1 \longrightarrow X$ along the submanifold M_f (in fact, a family of connections) with horizontal projector $\tilde{\mathbf{h}}$ verifying the boundary compatibility condition, and which is a solution of equation (9) and satisfies the boundary condition. M_f projects onto an open submanifold of X (and B^*_f projects also onto an open submanifold of ∂X).

If M_f is the final constraint submanifold and $j_{f1} : M_f \longrightarrow M_1$ is the canonical immersion then we may consider the $(n+2)$-form $\Omega_{M_f} = j^*_{f1}\Omega_{M_1}$, and the $(n+1)$-form $\Theta_{M_f} = i^*_{f1}\Theta_{M_1}$, where $\Omega_{M_f} = -d\Theta_{M_f}$.

Denoting by $leg_i := leg_L|_{Z_i}$, a direct computation shows that $leg_1(Z_a) = M_a$ for each integer.

$$
\begin{array}{lcll}
Z_1 = Z & \xrightarrow{\quad leg_1 \quad} & leg_L(Z) = M_1 & \xhookrightarrow{\quad j \quad} Z^* \\
\uparrow i_1 & & \uparrow j_1 & \\
Z_2 & \xrightarrow{\quad leg_2 \quad} & M_2 & \\
\uparrow i_2 & & \uparrow j_2 & \\
Z_3 & \xrightarrow{\quad leg_3 \quad} & M_3 & \\
\uparrow i_3 & & \uparrow j_3 & \\
\vdots & & \vdots & \\
\uparrow i_{k-2} & & \uparrow j_{k-2} & \\
Z_{k-1} & \xrightarrow{\quad leg_{k-1} \quad} & M_{k-1} & \\
\uparrow i_{k-1} & & \uparrow j_{k-1} & \\
Z_k & \xrightarrow{\quad leg_k \quad} & Z_k &
\end{array}
$$

In consequence, both algorithms have the same behaviour; in particular, if one of them stabilizes, so does the other, and at the same step. In particular, we have $leg_1(Z_f) = M_f$. In such a case, the restriction $leg_f : Z_f \longrightarrow M_f$ is a surjective submersion (that is, a fibration) and $leg_f^{-1}(leg_f(z)) = leg_1^{-1}(leg_1(z))$, for all $z \in Z_f$ (that is, its fibres are the ones of leg_1).

Therefore, the Lagrangian and Hamiltonian sides can be compared through the fibration $leg_f : Z_f \longrightarrow M_f$. Indeed, if we have a connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f with horizontal projector $\mathbf{h}$ which is a solution of equation (6) (the De Donder equations) and satisfies the boundary condition and, in addition, the connection is projectable via Leg_f to a connection in the fibration $\pi_{X\tilde{Z}} : \tilde{Z} \longrightarrow X$ along the submanifold M_f, then the horizontal projector of the projected connection is a solution of equation (8) (the Hamilton equations) and satisfies the boundary contion, too. Conversely, given a connection in the fibration $\pi_{X\tilde{Z}} : \tilde{Z} \longrightarrow X$ along the submanifold M_f, with horizontal projector $\tilde{\mathbf{h}}$ which is a solution of equation (8) satisfying the boundary condition, then every connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f that projects onto $\tilde{\mathbf{h}}$ is a solution of the De Donder equations (6) and satisfies the boundary condition.

4 Cartan formalism in the space of Cauchy data

4.1 Cauchy surfaces. Initial value problem

Definition 4.1. *A* ***Cauchy surface*** *is a pair (M, τ) formed by a compact oriented n-manifold M embedded in the base space X by $\tau : M \longrightarrow X$, such that $\tau(\partial M) \subseteq \partial X$, and the interior of M is included in the interior of X. Two of such Cauchy surfaces are considered the same up to an orientation and volume preserving diffeomorphism of M.*
In what follows, we shall fix M, and consider certain space $\tilde{X}$ of such embeddings. We shall rather call ***Cauchy surfaces*** *to such embeddings.*

The choice of M and $\tilde{X}$ depends on the physical theory which we aim to describe with this model.

Definition 4.2. *A* ***space of Cauchy data*** *is the manifold of embeddings $\gamma : M \to Z$ such that there exists a section ϕ of π_{XY} satisfying*

$$\gamma = (j^1\phi) \circ \tau$$

where $\tau := \pi_{XZ} \circ \gamma \in \tilde{X}$, and $\gamma(\partial M) \subseteq B$.
The space of such embeddings shall be denoted by $\tilde{Z}$, and we shall denote by $\pi_{\tilde{X}\tilde{Z}}$ the projection $\pi_{\tilde{X}\tilde{Z}}(\gamma) = \pi_{XZ} \circ \gamma$. We shall also require this projection to be a locally trivial fibration.

Definition 4.3. *The* ***space of Dirichlet data*** *is the manifold $\tilde{Y}$ of all the embeddings $\delta : M \longrightarrow Y$ of the form $\delta = \pi_{YZ} \circ \gamma$ for $\gamma \in \tilde{Z}$. We also define $\pi_{\tilde{Y}\tilde{Z}} : \tilde{Z} \longrightarrow \tilde{Y}$ as $\pi_{\tilde{Y}\tilde{Z}}(\gamma) = \pi_{YZ} \circ \gamma$.*
We denote by $\pi_{\tilde{X}\tilde{Y}}$ the unique mapping from $\tilde{Y}$ to $\tilde{X}$ such that $\pi_{\tilde{X}\tilde{Z}} = \pi_{\tilde{X}\tilde{Y}} \circ \pi_{\tilde{Y}\tilde{Z}}$ (see Figure 3)

A tangent vector v at $\gamma \in \tilde{Z}$ can be seen as a vector field along γ, that is, $v : M \longrightarrow TZ$ such that $\tau_Z \circ v = \gamma$, where $\tau_Z : TZ \longrightarrow Z$ is the canonical projection. Therefore, we identify vectors in $T_\gamma\tilde{Z}$ with vector fields on $\gamma(M)$. Thus, a vector field ξ_Z on Z induces a vector field $\xi_{\tilde{Z}}$ on $\tilde{Z}$, where for every $\gamma \in \tilde{Z}$, its representative tangent vector at $\gamma \in \tilde{Z}$ is given by

$$\xi_{\tilde{Z}}(\gamma)(u) = \xi_Z(\gamma(u))$$

for $u \in M$. And conversely, forms on Z can be considered to act upon tangent vectors of $\tilde{Z}$, for if $z = \gamma(u)$, α is a r-form on Z and $v \in T_\gamma\tilde{Z}$, then $\iota_v\alpha$ is a $(r-1)$-form on Z defined by

$$(\iota_v\alpha)_z := \iota_{v(u)}\alpha_z$$

In practice, no distinction between them will be made.

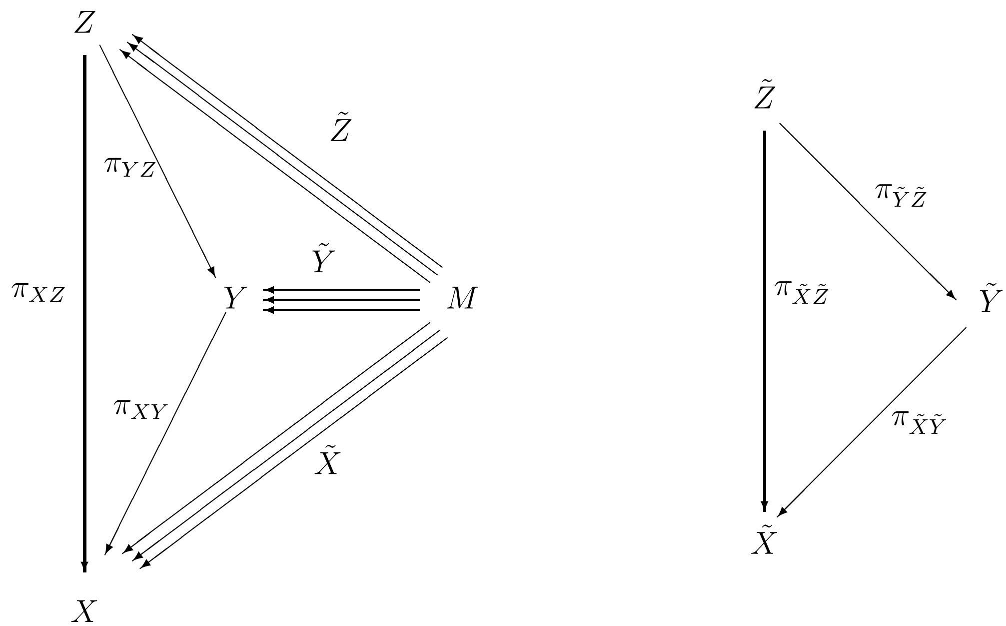

Figure 3

Integration gives a standard method for obtaining k-forms on $\tilde{Z}$ from $(k+n)$-forms on Z as follows.

Definition 4.4. *If α is a $(k+n)$-form in Z such that $i_B^*\alpha = d\beta$, we define the k-form $\tilde{\alpha}$ on $\tilde{Z}$ by*

$$\iota_{\tilde{\zeta}_1} \dots \iota_{\tilde{\zeta}_k} \tilde{\alpha}_\gamma = \int_M \gamma^* \iota_{\zeta_1} \dots \iota_{\zeta_k} \tilde{\alpha}_\alpha - (-1)^k \int_{\partial M} \gamma^* \iota_{\zeta_1} \dots \iota_{\zeta_k} \beta \tag{10}$$

for $\tilde{\zeta}_1, \dots, \tilde{\zeta}_k \in T_\gamma \tilde{Z}, \gamma \in \tilde{Z}$.

In particular, the Poincaré-Cartan $(n+1)$-form Θ_L and $(n+2)$-form Ω_L also induce a 1-form $\widetilde{\Theta_L}$ and a 2-form $\widetilde{\Omega_L}$ on $\tilde{Z}$, given by:

$$(\widetilde{\Theta_L})_\gamma(\tilde{\xi}) = \int_M \gamma^*(\iota_\xi \Theta_L) + \int_{\partial M} \gamma^*(\iota_\xi \Pi)$$

and also

$$\widetilde{\Omega_L}(\tilde{\xi}_1, \tilde{\xi}_2) = \int_M \gamma^*(\iota_{\xi_2}\iota_{\xi_1}\Omega_L).$$

Lemma 4.1. *If $\tilde{\xi}$ is a vector field on $\tilde{Z}$ defined from a vector field ξ on Z, and α is an n-form on Z such that $i_B^*\alpha = d\beta$ then*

$$d\tilde{\alpha}(\tilde{\xi})_\gamma = (\pounds_{\tilde{\xi}}\tilde{\alpha})_\gamma = \int_M \gamma^*(\pounds_\xi \alpha) - \int_{\partial M} \gamma^*(\pounds_\xi \beta)$$

Proof. First observe that $\tilde{\alpha}$ is a function. In this case, if $c_{\tilde{Z}}(t)$ is a curve such that $c_{\tilde{Z}}(0) = \gamma$ and $\dot{c}_{\tilde{Z}}(0) = \xi(\gamma)$, then

$$\begin{aligned} d\tilde{\alpha}(\tilde{\xi})_\gamma &= \tilde{\xi}_\gamma(\tilde{\alpha}) = \frac{d}{dt}(\tilde{\alpha} \circ c_{\tilde{Z}}(t))_{|t=0} = \frac{d}{dt}\left[\int_M (c_{\tilde{Z}}(t)^*\alpha) - \int_{\partial M} (c_{\tilde{Z}}(t)^*\beta)\right]_{|t=0} \\ &= \int_M \frac{d}{dt}\,(c_{\tilde{Z}}(t)^*\alpha)_{|t=0} - \int_{\partial M} \frac{d}{dt}\,(c_{\tilde{Z}}(t)^*\beta)_{|t=0} = \int_M \gamma^*(\pounds_\xi \alpha) - \int_{\partial M} \gamma^*(\pounds_\xi \beta). \quad \blacksquare \end{aligned}$$

The previous result can be also extended for forms of higher degree, and for arbitrary fibrations over X.

Let ξ be a complete vector field on a fibration W over X, and let us denote by $\tilde{W}$ certain space of embeddings in W, and by $\tilde{\xi}$ the vector field defined on $\tilde{W}$ from ξ (that is, $\tilde{\xi}(\gamma)(u) = \xi(\gamma(u))$).

Fix $\gamma \in \tilde{W}$. For every $u \in M$, consider an integral curve c^u of ξ through $\gamma(u)$, that is

$$\begin{aligned} c^u(0) &= \gamma(u) \\ \dot{c}^u(0) &= \xi(\gamma(u)) \end{aligned}$$

Let us define a curve $\tilde{c}$ on $\tilde{W}$ by

$$\tilde{c}(t)(u) = c^u(t).$$

Then we have that

Proposition 4.2. *$\tilde{c}$ is an integral curve of $\tilde{\xi}$ through γ.*

Proof. To see this, we just have to compute

$$\tilde{c}(0)(u) = c^u(0) = \gamma(u)$$

and

$$\dot{\tilde{c}}(0)(u) = \frac{d}{dt}(\tilde{c}(t))|_{t=0}(u) = \frac{d}{dt}(\tilde{c}(t)(u))|_{t=0} = \frac{d}{dt}c^u(t)|_{t=0} = \dot{c}^u(t) = \xi(\gamma(u)) = \tilde{\xi}(\gamma)(u). \quad \blacksquare$$

$\tilde{c}$ will be said to be the associated curve to the flow given by the c^u's.

In particular, if we also have a diffeomorphism $F : W \longrightarrow W$, it is easy to see that the curve (denoted by $\widetilde{F \circ c}$) associated to the family $F \circ c^u$ is precisely $\tilde{F} \circ \tilde{c}$.

To see this, and using the preceding notation, note first that

$$\widetilde{F \circ c}(t)(u) = (F \circ c)^u(t) = (F \circ c^u)(t) = F(c^u(t)) = F(\tilde{c}(t)(u)) = (\tilde{F} \circ \tilde{c}(t))(u),$$

from which we deduce

Corollary 4.3. *If $F : W \longrightarrow W$ is a diffeomorphism, then $T\tilde{F}(\tilde{\xi}) = \widetilde{TF(\xi)}$.*

The next step is to study the pullback of forms.

Proposition 4.4. *If $F : W \longrightarrow W$ is a diffeomorphism, and α is a $(n+k)$-form on W, such that $i_B^*\alpha = d\beta$, then*

$$\tilde{F}^*\tilde{\alpha} = \widetilde{F^*\alpha}$$

Proof. Let $\widetilde{V_1}, \ldots, \widetilde{V_k} \in T_{\tilde{F}^{-1}(\gamma)}\tilde{W}$. We have that

$$\begin{aligned}
\iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \tilde{F}^*\tilde{\alpha} &= \tilde{\alpha}(T\tilde{F}(\widetilde{V_1}), \ldots, T\tilde{F}(\widetilde{V_k})) = \tilde{\alpha}(\widetilde{TF(V_1)}, \ldots, \widetilde{TF(V_k)}) \\
&= \int_M \gamma^* \iota_{TF(V_1)} \ldots \iota_{TF(V_k)}\alpha - (-1)^k \int_{\partial M} \gamma^* \iota_{TF(V_1)} \ldots \iota_{TF(V_k)}\beta \\
&= \int_M (F^{-1} \circ \gamma)^* F^* \iota_{TF(V_1)} \ldots \iota_{TF(V_k)}\alpha - (-1)^k \int_{\partial M} (F^{-1} \circ \gamma)^* F^* \iota_{TF(V_1)} \ldots \iota_{TF(V_k)}\beta \\
&= \int_M (F^{-1} \circ \gamma)^* \iota_{V_1} \ldots \iota_{V_k} F^*\alpha - (-1)^k \int_{\partial M} (F^{-1} \circ \gamma)^* \iota_{V_1} \ldots \iota_{V_k} F^*\beta \\
&= \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \widetilde{F^*\alpha}.
\end{aligned}$$

∎

Finally,

Proposition 4.5. *If ξ is a vector field on $\tilde{W}$, then*

$$\pounds_{\tilde{\xi}}\tilde{\alpha} = \widetilde{\pounds_\xi \alpha}$$

Proof. Let $\widetilde{V_1}, \ldots, \widetilde{V_k} \in T_\gamma\tilde{W}$, and denote by ϕ_t the flow of ξ. Then we have that

$$\begin{aligned}
\iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \pounds_{\tilde{\xi}}\tilde{\alpha} &= \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \frac{d}{dt}\widetilde{\phi_t}^*\tilde{\alpha}|_{t=0} = \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \frac{d}{dt}\widetilde{\phi_t^*\alpha}|_{t=0} \\
&= \frac{d}{dt}\left(\iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \widetilde{\phi_t^*\alpha}\right)|_{t=0} = \frac{d}{dt}\left(\int_M \iota_{V_1} \ldots \iota_{V_k}\phi_t^*\alpha - (-1)^k \int_{\partial M} \iota_{V_1} \ldots \iota_{V_k}\phi_t^*\beta\right)|_{t=0} \\
&= \int_M \iota_{V_1} \ldots \iota_{V_k} \frac{d}{dt}\left(\phi_t^*\alpha\right)|_{t=0} - (-1)^k \int_{\partial M} \iota_{V_1} \ldots \iota_{V_k} \frac{d}{dt}\left(\phi_t^*\beta\right)|_{t=0} \\
&= \int_M \iota_{V_1} \ldots \iota_{V_k} \pounds_\xi\alpha - (-1)^k \int_{\partial M} \iota_{V_1} \ldots \iota_{V_k} \pounds_\xi\beta \\
&= \iota_{\widetilde{V_1}} \ldots \iota_{\widetilde{V_k}} \widetilde{\pounds_\xi\alpha}.
\end{aligned}$$

where for the last bit just notice that $i_B^*\pounds_\xi\alpha = \pounds_\xi i_B^*\alpha = \pounds_\xi d\beta = d\pounds_\xi\beta$. ∎

Back to the fibration $Z \longrightarrow X$, the consistency of our definition of forms respect to the exterior derivative is ensured by the following proposition

Proposition 4.6. *If α is an n-form or an $(n+1)$-form, then*

$$\widetilde{d\alpha} = d\tilde{\alpha}$$

In particular,

$$\widetilde{\Omega_L} := -d\widetilde{\Theta_L}$$

Proof. For n-forms we use the previous lemma

$$\begin{aligned}(d\tilde{\alpha})_\gamma(\tilde{\xi}) &= \int_M \gamma^* \pounds_\xi \alpha - \int_{\partial M} \gamma^* \pounds_\xi \beta = \int_M \gamma^* \iota_\xi d\alpha + \int_M \gamma^* d\iota_\xi \alpha - \int_{\partial M} \gamma^*(i_\xi d\beta + di_\xi \beta) \\ &= \int_M \gamma^* \iota_\xi d\alpha = (\widetilde{d\alpha})_\gamma(\xi)\end{aligned}$$

For $(n+1)$-forms:

$$\begin{aligned}d\widetilde{\alpha}(\xi,\zeta)_\gamma &= \{\xi(\widetilde{\alpha}(\zeta)) - \zeta(\widetilde{\alpha}(\xi)) - \widetilde{\alpha}([\zeta,\xi])\}_\gamma \\ &= \int_M \gamma^*\{\pounds_\xi(\iota_\zeta\alpha) - \pounds_\zeta(\iota_\xi\alpha) - \iota_{[\xi,\zeta]}\alpha\} \\ &+ \int_{\partial M} \gamma^*\{\pounds_\xi(\iota_\zeta\beta) - \pounds_\zeta(\iota_\xi\beta) - \iota_{[\xi,\zeta]}\beta\} \\ &= \int_M \gamma^*\{\iota_\zeta\iota_\xi d\alpha - d\iota_\zeta\iota_\xi\alpha\} \\ &+ \int_{\partial M} \gamma^*\{\iota_\zeta\iota_\xi d\beta - d\iota_\zeta\iota_\xi\beta\} \\ &= \int_M \gamma^*(\iota_\zeta\iota_\xi d\alpha) - \int_{\partial M} \gamma^*(\iota_\zeta\iota_\xi(d\beta - \alpha)) \\ &= \int_M \gamma^*(\iota_\zeta\iota_\xi d\alpha) \\ &= \widetilde{d\alpha}(\xi,\zeta)_\gamma.\end{aligned}$$

∎

4.2 The De Donder equations in the space of Cauchy data

The De Donder equations of Field Theories have a presymplectic counterpart in the spaces of Cauchy data. The relationship between both can be found in [3] (see also [28]), and requires the definition of a slicing of the base manifold X.

Definition 4.5. *We say that a curve $c_{\tilde{X}}$ in $\tilde{X}$ defined on a domain $I \subseteq \mathbb{R}$ **splits** X if the mapping $\Phi : I \times M \longrightarrow X$, such that $\Phi(t,u) = c_{\tilde{X}}(t)(u)$, is a diffeomorphism. In particular, the partial mapping $\Phi(t,\cdot)$ (defined by $\Phi(t,\cdot)(u) = \Phi(t,u)$) is an element of $\tilde{X}$ for all $t \in I$. In this case, $c_{\tilde{X}}$ is said to be a **slicing**.*

In this situation, we can rearrange coordinates in X such that if $\frac{\partial}{\partial t}$ generates the tangent space to I, then $T\Phi(\frac{\partial}{\partial t}) = \frac{\partial}{\partial x^0}$, and we consider $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ as local tangent vector fields on M or X.

Definition 4.6. *We can also define the concept of* ***infinitesimal slicing*** *at $\tau \in \tilde{X}$ as a tangent vector $v \in T_\tau \tilde{X}$ such that for every $u \in M$, $v(u)$ is transverse to $Im\, \tau$.*

If $c_{\tilde{Z}}$ is a curve in $\tilde{Z}$ such that its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X, then it defines a local section σ of π_{XZ} by

$$\sigma(c_{\tilde{X}}(t)(u)) = c_{\tilde{Z}}(t)(u) \tag{11}$$

Conversely, if σ is a section of π_{XZ}, and $c_{\tilde{X}}$ is a curve on $\tilde{X}$ (not necessarily a slicing), we define a curve $c_{\tilde{Z}}$ on Z by using (11). The following result relating equations in Z and $\tilde{Z}$ can be found in [3].

Theorem 4.7. *If σ satisfies the De Donder equations, then $c_{\tilde{Z}}$ defined as above verifies*

$$\iota_{\dot{c}_{\tilde{Z}}} \widetilde{\Omega_L} = 0 \tag{12}$$

Conversely, if $c_{\tilde{Z}}$ is a curve on Z satisfying (12), and its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X, then the section σ of π_{XZ} defined by (11) verifies the De Donder equations.

Proof. Assume that σ verifies the De Donder equations. From (11) we obtain that $\dot{c}_{\tilde{Z}} = \sigma_* \dot{c}_{\tilde{X}}$, whence

$$c_{\tilde{Z}}(t)^*(\iota_{\dot{c}_{\tilde{Z}}} \iota_\xi \Omega_L) = c_{\tilde{X}}(t)^* \sigma^*(\iota_{\dot{c}_{\tilde{Z}}} \iota_\xi \Omega_L) = c_{\tilde{X}}(t)^*(\iota_{\dot{c}_{\tilde{X}}} \sigma^* \iota_\xi \Omega_L) = 0$$

for all ξ. Now integrate over M to obtain the desired result. For the converse, consider the integral

$$0 = \int_M c_{\tilde{X}}(t)^*(\iota_{\dot{c}_{\tilde{X}}} \sigma^* \iota_\xi \Omega_L) = 0$$

since this is true for every ξ, from the Fundamental Theorem of Calculus of Variations, we deduce

$$c_{\tilde{X}}(t)^*(\iota_{\dot{c}_{\tilde{X}}} \sigma^* \iota_\xi \Omega_L) = 0$$

Now if $c_{\tilde{X}}$ splits X, then $\dot{c}_{\tilde{X}}(t)$ is transverse to $c_{\tilde{X}}(t)(M)$, which implies the De Donder equations. ∎

Note that, in particular, if $\mathbf{h}$ is the horizontal projector of a connection which is a solution of the De Donder equations for a connection

$$\iota_{\mathbf{h}} \Omega_L = n \Omega_L \tag{13}$$

and if σ is a horizontal local section of $\mathbf{h}$, the results above show that the solution to (12) is the horizontal lift of $\dot{c}_{\tilde{X}}$ through $\mathbf{h}$. Or more generally, the solutions are obtained as horizontal lifts of infinitesimal slicings through the connection solution to (13).

4.3 The singular case

For a singular Lagrangian, we cannot guarantee the existence of a curve $c_{\tilde{Z}}$ in $\tilde{Z}$ as a solution of the De Donder equations in $\tilde{Z}$.

Therefore, we propose an algorithm similar to that of a general presymplectic space (developed in [26, 30, 31]; see also [8, 45, 47] for the time dependent case), where to the condition that defines the manifold obtained in each step (which is the existence of a tangent vector verifying the De Donder equations), we add the fact that this tangent vector must project onto an infinitesimal slicing.

Naming $\tilde{Z}_1 := \tilde{Z}$, we define $\tilde{Z}_2$ and the subsequent subsets (requiring them to be submanifolds) as follows

$$\begin{aligned}
&\tilde{Z}_2 := \{\gamma \in \tilde{Z}_1 | \exists v \in T_\gamma \tilde{Z}_1 \text{ such that } T\pi_{\tilde{X}\tilde{Z}}(v) \text{ is an infinitesimal slicing and } \iota_v \widetilde{\Omega_L}|_\gamma = 0\} \\
&\tilde{Z}_3 := \{\gamma \in \tilde{Z}_2 | \exists v \in T_\gamma \tilde{Z}_2 \text{ such that } T\pi_{\tilde{X}\tilde{Z}}(v) \text{ is an infinitesimal slicing and } \iota_v \widetilde{\Omega_L}|_\gamma = 0\} \\
&\quad \ldots
\end{aligned}$$

In the favourable case, the algorithm will stop at certain final non-zero dimensional constraint submanifold $\tilde{Z}_f$.

This algorithm is closely related to the algorithm in the finite dimensional spaces. We turn now to state the link between them.

Proposition 4.8. *Suppose that we have $v \in T_\gamma \tilde{Z}_1$ such that $T\pi_{\tilde{X}\tilde{Z}}(v)$ is an infinitesimal slicing and $\iota_v \widetilde{\Omega_L}|_\gamma = 0$. Then, for every $u \in M$ we have that*

$$H_{\gamma(u)} := T_u\gamma(T_uM) \oplus \langle v(u) \rangle$$

is a horizontal subspace of $T_{\gamma(u)}Z$ which horizontal projector $\boldsymbol{h}$ verifies the De Donder equations for connections satisfying (13) *at $\gamma(u)$:*

$$\iota_{\mathbf{h}} \Omega_L|_{\gamma(u)} = n\Omega_L|_{\gamma(u)}$$

Proof. The fact that v projects onto an infinitesimal slicing guarantees that $H_{\gamma(u)}$ is indeed horizontal.

The other hypothesis states that

$$\gamma^*(\iota_\xi \iota_{v_{\gamma(u)}} \Omega_L) = 0$$

for every $\xi \in T_{\gamma(u)}Z$, that is, if $\langle v_1, v_2, \ldots, v_n \rangle$ is a basis for T_uM, then

$$\iota_\xi \iota_{v_{\gamma(u)}} \Omega_L(T_u\gamma(v_1), T_u\gamma(v_2), \ldots, T_u\gamma(v_n)) = 0$$

or in other words,

$$\Omega_L(\xi, H_1, H_2, \ldots, H_{n+1}) = 0$$

for every $\xi \in T_{\gamma(u)}Z$ and every collection $H_1, H_2, \ldots, H_{n+1}$ of horizontal tangent vectors.

We want to prove that $\iota_{\mathbf{h}}\Omega_L|_{\gamma(u)} = n\Omega_L|_{\gamma(u)}$, or equivalently, $\iota_\xi\iota_{\mathbf{h}}\Omega_L|_{\gamma(u)} = n\iota_\xi\Omega_L|_{\gamma(u)}$, for every $\xi \in T_{\gamma(u)}Z$.

From the previous remarks, we see that the condition results to be true when it is evaluated on $n+1$ horizontal vector fields.

Suppose that V_1 is a vertical tangent vector to $\gamma(u)$. Then (as $\mathbf{h}(V_1) = 0$),

$$\iota_{\mathbf{h}}\Omega_L(\xi, V_1, H_1, \ldots, H_n) \quad = \quad \Omega_L(\mathbf{h}(\xi), V_1, H_1, \ldots, H_n) + n\Omega_L(\xi, V_1, H_1, \ldots, H_n)$$

where the first term vanishes due to the previous remarks. Thus, the expression holds when applied to any two tangent vector, and to any n horizontal tangent vectors.

For the next step, having two vertical vectors, remember that Ω_L is annihilated by three vertical tangent vectors. Therefore,

$$\begin{aligned}
\iota_{\mathbf{h}}\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) &= \Omega_L(\mathbf{h}(\xi), V_1, V_2, H_1, \ldots, H_{n-1}) \\
&\quad + (n-1)\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) \\
&= \Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) + (n-1)\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1}) \\
&= n\Omega_L(\xi, V_1, V_2, H_1, \ldots, H_{n-1})
\end{aligned}$$

Finally, from the mentioned properties of Ω_L, the expression also holds for a higher number of vertical tangent vectors, and so the expression holds in general. ∎

As an immediate result, we have that

Corollary 4.9. *If $\gamma \in \tilde{Z}_2$, then $Im\gamma \subseteq Z_2$.*

and in general,

Proposition 4.10. *If $\gamma \in \tilde{Z}_i$, then $Im\gamma \subseteq Z_i$.*

Proof. If $\gamma \in \tilde{Z}_i$ (which implies that there exists $v \in T\tilde{Z}_i$ such that $\iota_v\widetilde{\Omega_L}|_\gamma = 0$), then for every $u \in M$ we define $H_{\gamma(u)} := T\gamma_u(T_uM) \oplus \langle v(u) \rangle$.

We need to justify in each step that $H_{\gamma(u)} \subseteq T_{\gamma(u)}Z_i$, which amounts to prove that $T\gamma_u(T_uM) \subseteq T_{\gamma(u)}Z_i$ and $v(u) \in T_{\gamma(u)}Z_i$. The first assertion is true by construction of the subsets.

To see that $v(u) \in T_{\gamma(u)}Z_i$, we proceed inductively, starting on $i = 2$, for which the result is true because of the preceding corollary.

We assume it to be true for all the steps until the i-th, and we prove that $v(u) \in T_{\gamma(u)}Z_{i+1}$.

As $\gamma \in \tilde{Z}_{i+1}$, there exists $v \in T_\gamma\tilde{Z}_i$ such that $\iota_v\widetilde{\Omega_L} = 0$. Thus, there exists a curve $c : (-\varepsilon, \varepsilon) \longrightarrow \tilde{Z}_i$ (and thus $Im(c)(t) \subseteq Z_i$) such that $c(0) = \gamma$ and $\dot{c}(o) = v$. We deduce that $v(u) \in T_{\gamma(u)}Z_i$. ∎

Remark 4.11. *Suppose now that $\tilde{X}$ admits an slicing. In the case in which $z \in Z_i$ is such that $\pi_{XZ}(z)$ belongs to the image of the slicing, and $\boldsymbol{h}_z$ is integrable, then there exists $\gamma \in \tilde{Z}_i$, and $u \in M$ such that $\gamma(u) = z$.*

As before, we prove first the case $i = 2$. If σ is an horizontal local section of $\boldsymbol{h}$ at z, then we use the slicing to define the curve $c_{\tilde{Z}}(t)$, which verifies the De Donder equations in $\tilde{Z}$, and projects onto the slicing, therefore we can take $\gamma = c_{\tilde{Z}}(t)$ for some t.

For the case $i > 1$, simply observe that if $H_{\gamma(u)} \subseteq Z_i$, then $\dot{c}_{\tilde{Z}}(t)(u')$ must be tangent to Z_2 for all $u' \in M$, and a very similar argument to that of the preceding section proves that $\gamma = c_{\tilde{Z}}(t) \in \tilde{Z}_2$.

4.4 Brackets

Notice that, in general, the only fact over $\widetilde{\Omega_L}$ that we can guarantee is that it is presymplectic, as we cannot guarantee nor the existence neither the uniqueness of Hamiltonian vector fields associated to functions defined on $\tilde{Z}$. For further details see [50] and [51].

Definition 4.7. *Given a function f in $\tilde{Z}$ and a vector field $\tilde{\xi}$ on $\tilde{Z}$, we shall say that f is a* ***Hamiltonian function****, and that $\tilde{\xi}$ is a* ***Hamiltonian vector field*** *for f if*

$$\iota_{\tilde{\xi}}\widetilde{\Omega_L} = df$$

Proposition 4.12. *If α is a Hamiltonian n-form in Z for Ω_L which is exact on ∂Z, say $\tilde{\alpha}_{|\partial Z} = d\tilde{\beta}$, then $\tilde{\alpha}$ is a Hamiltonian function on $\tilde{Z}$ for $\widetilde{\Omega_L}$. More precisely, if X_α is a Hamiltonian vector field for α, then $X_{\tilde{\alpha}}$ defined on $\tilde{Z}$ by*

$$[X_{\tilde{\alpha}}(\gamma)](u) = X_\alpha(\gamma(u))$$

is a Hamiltonian vector field for $\tilde{\alpha}$

Proof. Take a tangent vector $\tilde{\xi}$ to $\tilde{Z}$, then by lemma (4.1)

$$\begin{aligned}
(d\tilde{\alpha})(\tilde{\xi})|_\gamma &= \int_M \gamma^*(\pounds_\xi \alpha) - \int_{\partial M} \gamma^*(\pounds_\xi \beta) \\
&= \int_M \gamma^* \iota_\xi d\alpha + \int_M \gamma^* d\iota_\xi \alpha - \int_{\partial M} \gamma^* \iota_\xi d\beta \\
&= \int_M \gamma^* \iota_\xi d\alpha = \int_M \gamma^* \iota_\xi \iota_{X_\alpha} \Omega_L = \iota_{\tilde{X}_\alpha} \widetilde{\Omega_L}(\tilde{\xi})|_\gamma.
\end{aligned}$$

which proves that $d\tilde{\alpha} = \iota_{X_{\tilde{\alpha}}} \widetilde{\Omega_L}$. ∎

If f is a Hamiltonian function on $\tilde{Z}$, then its associated Hamiltonian vector field is defined up to an element in the kernel of $\widetilde{\Omega_L}$, therefore we can define the bracket operation for these functions as follows.

Definition 4.8. *If f and g are Hamiltonian functions on $\tilde{Z}$, with associated Hamiltonian vector fields X_f and X_g, then we define:*

$$\{f,g\} := \widetilde{\Omega_L}(X_f, X_g)$$

Notice that $i_B^*\Omega_L = 0$, thus if α_1 and α_2 are Hamiltonian forms which are exact on the boundary, then $i_B^*\{\alpha_1,\alpha_2\} = 0$.

Proposition 4.13. *If α_1 and α_2 are Hamiltonian n-forms which are exact on ∂Z, then*

$$\{\tilde{\alpha_1}, \tilde{\alpha_2}\} = \widetilde{\{\alpha_1, \alpha_2\}}$$

Proof.

$$\{\tilde{\alpha_1}, \tilde{\alpha_2}\} = \widetilde{\Omega_L}(X_{\tilde{\alpha_1}}, X_{\tilde{\alpha_2}}) = \int_M \gamma^* \iota_{X_{\alpha_2}} \iota_{X_{\alpha_1}} \Omega_L = \int_M \gamma^* \{\alpha_1, \alpha_2\} = \widetilde{\{\alpha_1, \alpha_2\}}.$$

∎

In [6, 19] and [25] the authors explore the properties of a generalisation of this bracket, which satisfies the graded versions of several properties, such as skew-symmetry and Jacobi identity.

Remark 4.14. *We could alternatively use the space of Cauchy data $\tilde{Z}^*$, defined in the obvious way. But nothing different or new would be obtained. In fact, assume for simplicity that L is hyperregular. Then we would have a diffeomorphism $\widetilde{leg_L} : \tilde{Z} \longrightarrow \tilde{Z}^*$ defined by composition:*

$$\widetilde{leg_L}(\gamma) = leg_L \circ \gamma$$

for all $\gamma \in \tilde{Z}$.

If the Lagrangian is not regular, but at least is almost regular, we invite to the reader to develop the corresponding scheme. The only delicate point is that we have to consider the second order problem in the Lagrangian side, so that $\widetilde{leg_L} : \tilde{Z} \longrightarrow \tilde{Z}^*$ becomes a fibration.

In what follows, we shall emphasize the discussion in the Lagrangian side, since, as we have shown, the equivalence with the Hamiltonian side is obvious.

5 Symmetries. Noether's theorems

We are now interested in studying the presence of symmetries which would eventually produce preserved quantities, and allow us to reduce the complexity of the dynamical system and to obtain valuable information about its behaviour. For every type of symmetry, there will be a form of the Noether's theorem, which will show up the preserved quantity obtained from it (see [60]).

We shall suppose that we are in the regular Lagrangian case, unless stated otherwise.

In our framework for field theory, we define a preserved quantity in the following manner:

Definition 5.1. *A* ***preserved quantity for the Euler-Lagrange equations*** *is an n-form α on Z such that $(j^1\phi)^* d\alpha = 0$ for every solution ϕ of the Euler-Lagrange equations. If α is a preserved quantity, then $\tilde{\alpha}$ is called its associated* ***momentum***.

Notice that if α is a preserved quantity, and Λ is a closed form, then $\alpha + \Lambda$ is also a preserved quantity. Similarly, if γ is an n-form which belongs to the differential ideal $\mathcal{I}(\mathcal{C})$, then $\alpha + \gamma$ is also a preserved quantity (see [60] for a further discussion).

We turn now to obtain preserved quantities from symmetries.

5.1 Symmetries of the Lagrangian

We shall define the notion of symmetry based on the the variation of the Poincaré-Cartan $(n+1)$-form along prolongations of vector fields. Suppose that ξ_Y is a vector field defined on Y, and abbreviate by F the function such that

$$\pounds_{\xi_Y^{(1)}} \mathcal{L} - F\eta \in \mathcal{I}(\mathcal{C})$$

having local expression

$$F = \xi_Y^{(1)}(L) + \left(\frac{\partial \xi_Y^\mu}{\partial x^\mu} + z_\nu^i \frac{\partial \xi_Y^\nu}{\partial y^i} \right) L. \tag{14}$$

After a lengthy computation we get that

$$\begin{aligned} \pounds_{\xi_Y^{(1)}} \Theta_L = {} & F\eta + \frac{\partial F}{\partial z_\mu^i} \theta^i \wedge d^n x_\mu \\ & + z_\nu^j \left(\frac{\partial \xi_Y^\nu}{\partial y^j} \frac{\partial L}{\partial z_\mu^i} - \frac{\partial \xi_Y^\mu}{\partial y^j} \frac{\partial L}{\partial z_\nu^i} \right) \theta^i \wedge d^n x_\mu \\ & - \frac{\partial \xi_Y^\nu}{\partial y^j} \frac{\partial L}{\partial z_\mu^i} \theta^i \wedge dy^j \wedge d^{n-1} x_{\nu\mu} \end{aligned} \tag{15}$$

Definition 5.2. *A vector field ξ_Y on Y is said to be an* ***infinitesimal symmetry of the Lagrangian*** *or a* ***variational symmetry*** *if $\pounds_{\xi_Y^{(1)}} \Theta_L \in \mathcal{I}(\mathcal{C})$ (the differential ideal generated by the contact forms), and $\xi_Y^{(1)}$ is also tangent to B and verifies $\pounds_{\xi_Y^{(1)}|_B} \Pi = 0$*

We shall only deal with infinitesimal symmetries of the Lagrangian, so for brevity they will be referred simply as symmetries of the Lagrangian.

From the definition and the expression (15), it is obvious to see that

Proposition 5.1. *If a vector field ξ_Y on Y is a symmetry of the Lagrangian, then $F = 0$ (where F was defined in* (14)*).*

Remark 5.2. *In our construction, we choose as definition of the Poincaré-Cartan* $(n+1)$*-form:*

$$\Theta_L = \mathcal{L} + (S_\eta)^*(dL)$$

or, in fibred coordinates

$$\Theta_L = L\, d^{n+1}x + \frac{\partial L}{\partial z^i_\mu}\theta^i \wedge d^n x_\mu$$

If $n > 0$ *it is possible to generalize the construction of the Poincaré-Cartan* $(n+1)$*-form in several different ways. The unique requirement is that the resulting* π_{YZ}*-semibasic* $(n+1)$*-form be* Lepage*-equivalent to* $\mathcal{L}$*, that is,*

$$\Theta - \mathcal{L} \in \mathcal{I}(\mathcal{C})$$

and $i_V d\Theta \in \mathcal{I}(\mathcal{C})$ *where* V *is an arbitrary* π_{YZ}*-vertical vector field. Locally,*

$$\Theta = \Theta_L + \cdots \tag{16}$$

where the dots signify terms which are at least two-contact (see [2, 10, 39, 43]). Obviously, all them gives us identically the same Euler-Lagrange equations.

Therefore, we may substitute in Definitions 5.2, 5.3 and 5.4 the Poincaré-Cartan $(n+1)$*-form by any* $(n+1)$*-form which is* Lepage- *equivalent to* Θ_L*. Obviously, the symmetries of the Euler-Lagrange equations are independent of the class of* Lepagian $(n+1)$*-form appearing in their definition.*

We also have the following two special cases, which are easily computed from the expression of F.

Proposition 5.3. *If* ξ_Y *is a projectable symmetry of the Lagrangian (*$T\pi_{XY}(\xi_Y)$ *is a well defined vector field, or locally* $\frac{\partial \xi^\mu_Y}{\partial y^i} = 0$*), or if* $dim X = 1$ $(n = 0)$*, then*

$$\pounds_{\xi^{(1)}_Y}\Theta_L = 0$$

or, equivalently,

$$\pounds_{\xi^{(1)}_Y}\mathcal{L} = 0$$

Therefore,

$$\xi^{(1)}_Y(L) = -\sum_\mu \frac{d\xi^\mu_Y}{dx^\mu}L$$

And as a direct consequence of Proposition 2.3, we have

Proposition 5.4. *The symmetries of the Lagrangian form a Lie subalgebra of* $\mathfrak{X}(Y)$*.*

Theorem 5.5. ***(Noether's theorem)****. If* ξ_Y *is a symmetry of the Lagrangian, then* $\iota_{\xi^{(1)}_Y}\Theta_L$ *is a preserved quantity, which is exact on the boundary.*

Proof. We have that

$$\pounds_{\xi_Y^{(1)}}\Theta_L = -\iota_{\xi_Y^{(1)}}\Omega_L + d\iota_{\xi_Y^{(1)}}\Theta_L$$

If ϕ is a solution of the Euler-Lagrange equations, then

$$0 = (j^1\phi)^*\pounds_{\xi_Y^{(1)}}\Theta_L = -(j^1\phi)^*\iota_{\xi_Y^{(1)}}\Omega_L + (j^1\phi)^* d\iota_{\xi_Y^{(1)}}\Theta_L,$$

where the first term vanishes by the intrinsic Euler-Lagrange equations (see Proposition 2.10).

Finally, to see that it is exact on the boundary, notice that from the boundary property of a symmetry of the Lagrangian we infer that $\iota_{\xi_Y^{(1)}{}_{|B}} d\Pi = -d\iota_{\xi_Y^{(1)}{}_{|B}}\Pi$, and from this we get

$$i_B^*(\iota_{\xi_Y^{(1)}}\Theta_L) = \iota_{\xi_Y^{(1)}{}_{|B}} d\Pi = -d\iota_{\xi_Y^{(1)}{}_{|B}}\Pi$$

∎

Observe that without the boundary condition, we obtain that $(j^1\phi)^* d\iota_{\xi_Y^{(1)}}\Theta_L = 0$, but we cannot be sure that it is exact on the boundary.

The preserved quantity can be written in local coordinates as

$$\left(\left[L - z_\mu^i \frac{\partial L}{\partial z_\mu^i}\right]\xi_X^\nu + \frac{\partial L}{\partial z_\nu^i}\xi_Y^i\right) d^n x_\nu - \frac{\partial L}{\partial z_\mu^i}\xi_X^\nu dy^i \wedge d^{n-1}x_{\mu\nu}$$

5.2 Noether symmetries

Definition 5.3. *A vector field ξ_Y on Y is said to be a* ***Noether symmetry*** *or a* ***divergence symmetry*** *if there exists an n-form on Y whose pullback α to Z (that must be exact $\alpha = d\beta$ on B) verifies $\pounds_{\xi_Y^{(1)}}\Theta_L - d\alpha \in \mathcal{I}(\mathcal{C})$, and $\xi_Y^{(1)}$ is tangent to B and verifies $\pounds_{\xi_Y^{(1)}{}_{|B}}\Pi = 0$*

The relation $dy^i = \theta^i + z_\mu^i dx^\mu$ allows us to write α locally as follows

$$\alpha = \alpha_\mu dx^0 \wedge \ldots \wedge \widehat{dx^\mu} \wedge \ldots \wedge dx^n + \theta$$

for $\theta \in \mathcal{I}(\mathcal{C})$ and

$$d\alpha - \sum_\mu (\frac{\partial \alpha^\mu}{\partial x^\mu} + z_\mu^i \frac{\partial \alpha^\mu}{\partial y^i})\eta \in \mathcal{I}(\mathcal{C})$$

Therefore, if we define:

$$\tilde{F} = F + \sum_\mu \left(\frac{\partial \alpha^\mu}{\partial x^\mu} + z_\mu^i \frac{\partial \alpha^\mu}{\partial y^i}\right)$$

then

Proposition 5.6. *If a vector field ξ_Y on Y is a Noether symmetry then $\tilde{F} = 0$.*

Similarly,

Proposition 5.7. *(1) If ξ_Y is a $\pi_{XY}-$projectable Noether symmetry, then*

$$\pounds_{\xi_Y^{(1)}} \Theta_L = d\alpha$$

Furthermore,

$$\xi_Y^{(1)}(L) = -\sum_\mu \left(\frac{d\xi_Y^\mu}{dx^\mu} L + \frac{d\alpha^\mu}{dx^\mu} \right)$$

(2) If $dimX = 1$ and ξ_Y is a Noether symmetry then

$$\pounds_{\xi_Y^{(1)}} \Theta_L = d\alpha$$

Proposition 5.8. *Noether symmetries form a Lie subalgebra of $\mathfrak{X}(Y)$, containing the Lie algebra of the symmetries of the Lagrangian.*

Proof.

$$\begin{aligned} \pounds_{[\xi_Y^{(1)}, \zeta_Y^{(1)}]} \Theta_L &= \pounds_{\xi_Y^{(1)}} \pounds_{\zeta_Y^{(1)}} \Theta_L - \pounds_{\zeta_Y^{(1)}} \pounds_{\xi_Y^{(1)}} \Theta_L = \pounds_{\xi_Y^{(1)}} (d\alpha_2 + \theta_2) - \pounds_{\zeta_Y^{(1)}} (d\alpha_1 + \theta_1) \\ &= d(\pounds_{\xi_Y^{(1)}} \alpha_2 - \pounds_{\zeta_Y^{(1)}} \alpha_1) + \pounds_{\xi_Y^{(1)}} \theta_2 - \pounds_{\zeta_Y^{(1)}} \theta_1 \end{aligned}$$

and $\pounds_{\xi_Y^{(1)}} \theta_2 - \pounds_{\zeta_Y^{(1)}} \theta_1 \in \mathcal{I}(\mathcal{C})$.

Finally, since $\xi_Y^{(1)}$ and $\zeta_Y^{(1)}$ are tangent to B, then $[\xi_Y^{(1)}, \zeta_Y^{(1)}]$ is also tangent to B. We also have that $\pounds_{[\xi_Y^{(1)}, \zeta_Y^{(1)}]_{|B}} \Pi = \pounds_{\xi_Y^{(1)}{}_{|B}} \pounds_{\zeta_Y^{(1)}{}_{|B}} \Pi - \pounds_{\zeta_Y^{(1)}{}_{|B}} \pounds_{\xi_Y^{(1)}{}_{|B}} \Pi = 0$ on B, and that if α_1 and α_2 are exact on B, so is $\pounds_{\xi_Y^{(1)}{}_{|B}} \alpha_2 - \pounds_{\zeta_Y^{(1)}{}_{|B}} \alpha_1$. ∎

The following Noether's theorem

Theorem 5.9. ***(Noether's theorem)**. If ξ_Y is a Noether symmetry, then $\iota_{\xi_Y^{(1)}} \Theta_L - \alpha$ is a preserved quantity which is exact on the boundary.*

is proved analogously as we did for the symmetries of the Lagrangian. We just remark a slight modification introduced to see that it is exact on the boundary:

$$i_B^*(\iota_{\xi_Y^{(1)}} \Theta_L - \alpha) = \iota_{\xi_Y^{(1)}{}_{|B}} d\Pi - d\beta = d(-\iota_{\xi_Y^{(1)}{}_{|B}} \Pi - \beta)$$

5.3 Cartan symmetries

Definition 5.4. *A vector field ξ_Z on Z is said to be a **Cartan symmetry** if its flow preserves the differential ideal $\mathcal{I}(\mathcal{C})$ (in other words, $\psi^*_{Z,t}\theta^i \in \mathcal{I}(\mathcal{C})$, or locally, $\pounds_{\xi_Z}\mathcal{I}(\mathcal{C}) \subseteq \mathcal{I}(\mathcal{C})$), and there exists an n-form α on Z (that must be exact $\alpha = d\beta$ on B) such that $\pounds_{\xi_Z}\Theta_L - d\alpha \in \mathcal{I}(\mathcal{C})$, ξ_Z is tangent to B and verifies $\pounds_{\xi_Z|_B}\Pi = 0$.*

If ξ_Y is a Noether symmetry, then its 1-jet prolongation is a Cartan symmetry. Conversely, it is obvious that a projectable Cartan symmetry is the 1-jet prolongation of its projection, which is therefore a Noether symmetry.

Proposition 5.10. *The Cartan symmetries form a subalgebra of $\mathfrak{X}(Z)$.*

We also have

Theorem 5.11. ***(Noether's theorem)**. If ξ_Z is a Cartan symmetry, then $\iota_{\xi_Z}\Theta_L - \alpha$ is a preserved quantity which is exact on the boundary.*

We also have the obvious relations between the different types of symmetries that we have exposed here. Every symmetry of the Lagrangian is a Noether symmetry. And the 1-jet prolongation of any Noether symmetry is a Cartan symmetry.

And finally,

Proposition 5.12. *The flow of Cartan symmetries maps solutions of the Euler-Lagrange equations into solutions of the Euler-Lagrange equations.*

Proof. Let ψ^t_Z be the flow of a Cartan symmetry ξ_Z.

For any section $\phi \in \Gamma(\pi)$, we can locally define

$$\psi^t_{\phi,X} := \pi_{XZ} \circ \psi^t_Z \circ j^1\phi$$

$\psi^0_{\phi,X} = Id_X$, whence for small $t's$, $\psi^t_{\phi,X}$ is a diffeomorphism. Analogously, we define

$$\psi^t_{\phi,Y} := \pi_{YZ} \circ \psi^t_Z \circ j^1\phi \circ \pi_{XY}$$

With the same argument we see that for small $t's$, $\psi^t_{\phi,Y}$ is as well a diffeomorphism.

If ϕ is a solution of the Euler-Lagrange equation, then the flow transforms ϕ into

$$\psi^t_{\phi,Y} \circ \phi \circ (\psi^t_{\phi,X})^{-1}$$

Now, for $\theta \in \mathcal{C}$,

$$(\psi^t_Z \circ j^1\phi \circ (\psi^t_{\phi,X})^{-1})^*\theta = ((\psi^t_{\phi,X})^{-1})^*(j^1\phi)^*(\psi^t_Z)^*\theta = 0$$

as ξ_Z is a Cartan symmetry. This means that $\psi^t_Z \circ j^1\phi \circ (\psi^t_{\phi,X})^{-1}$ is the 1-jet prolongation of its projection to Y,

$$\pi_{YZ} \circ \psi^t_Z \circ j^1\phi \circ (\psi^t_{\phi,X})^{-1} = \psi^t_{\phi,Y} \circ \phi \circ (\psi^t_{\phi,X})^{-1}$$

In other words,

$$j^1(\psi^t_{\phi,Y} \circ \phi \circ (\psi^t_{\phi,X})^{-1}) = \psi^t_Z \circ j^1\phi \circ (\psi^t_{\phi,X})^{-1}$$

Now we need to see that the transformed solution verifies the Euler-Lagrange equations. The preceding equation shows that, being the symmetry tangent to B, the boundary condition will be satisfied.

In addition, for every compact $(n+1)$-dimensional submanifold C, and every vertical vector field $\xi \in \mathcal{V}(\pi)$, which annihilates at ∂C (and therefore, so does $\xi^{(1)}$),

$$\begin{aligned}
&\int_{(\psi^t_{\phi,X})(C)} (j^1(\psi^t_{\phi,Y} \circ \phi \circ (\psi^t_{\phi,X})^{-1}))^* \pounds_{\xi^{(1)}} \Theta_L \\
&= \int_{(\psi^t_{\phi,X})(C)} (\psi^t_Z \circ j^1\phi \circ (\psi^t_{\phi,X})^{-1})^* \pounds_{\xi^{(1)}} \Theta_L \\
&= \int_C (\psi^t_Z \circ j^1\phi)^* \pounds_{\xi^{(1)}} \Theta_L = \int_C (j^1\phi)^* (\psi^t_Z)^* \pounds_{\xi^{(1)}} \Theta_L
\end{aligned}$$

by means of a change of variable. The annihilation of the preceding expression is infinitesimally equivalent to the annihilation of

$$\begin{aligned}
&\int_C (j^1\phi)^* \pounds_{\xi_Z} \pounds_{\xi^{(1)}} \Theta_L \\
&= \int_C (j^1\phi)^* \pounds_{[\xi_Z, \xi^{(1)}]} \Theta_L - \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} \pounds_{\xi_Z} \Theta_L
\end{aligned}$$

and we conclude by seeing that

$$\int_C (j^1\phi)^* \pounds_{[\xi_Z, \xi^{(1)}]} \Theta_L = -\int_C (j^1\phi)^* \iota_{[\xi_Z, \xi^{(1)}]} \Omega_L + \int_C (j^1\phi)^* d\iota_{[\xi_Z, \xi^{(1)}]} \Theta_L = 0$$

where the first term vanishes because ϕ is a solution of Euler-Lagrange equations, and second term vanishes due to the boundary condition on ξ; and

$$\begin{aligned}
\int_C (j^1\phi)^* \pounds_{\xi^{(1)}} \pounds_{\xi_Z} \Theta_L &= \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} (d\alpha + \theta) \\
&= \int_{\partial C} (j^1\phi)^* \pounds_{\xi^{(1)}} \alpha + \int_C (j^1\phi)^* \pounds_{\xi^{(1)}} \theta = 0
\end{aligned}$$

where the first term vanishes again by the boundary condition on ξ. ∎

5.4 Symmetries for the De Donder equations

In the discussion of the preceding section, we have used on Noether's theorem the fact that, for a solution ϕ of the Euler-Lagrange equations, we have

$$(j^1\phi)^*\theta = 0$$

for elements θ of the differential ideal generated by the contact forms. However, this result is no longer true for general solutions of the De Donder equations (more specifically, when the Lagrangian is not regular). In other words, if σ is a solution of the De Donder equations, then **not necessarily**

$$\sigma^*\theta = 0$$

for $\theta \in \mathcal{I}(\mathcal{C})$.

Therefore, our definition of symmetry must be more restrictive when we are dealing with solutions of the De Donder equations.

Definition 5.5. *A **preserved quantity for the De Donder equations** is a n-form α on Z such that $\sigma^* d\alpha = 0$ for every solution σ of the De Donder equations. If α is a preserved quantity, then $\tilde{\alpha}$ is called its associated **momentum**.*

Also note that if α is a preserved quantity and β is a closed n-form, then $\alpha + \beta$ is also a preserved quantity.

From equation (7) we can easily deduce the following.

Proposition 5.13. *Let* $\mathbf{h}$ *be a solution of the connection equation* (6)*. Then α is a preserved quantity for the De Donder equations if and only if $d\alpha$ is annihilated by any n horizontal tangent vectors at each point.*

Definition 5.6. *We have the following definitions of symmetries for the De Donder equations:*

*(1) A vector field ξ_Y on Y is said to be a **symmetry of the Lagrangian**, or a **variational symmetry** if*

$$\pounds_{\xi_Y^{(1)}} \Theta_L = 0$$

and $\xi_Y^{(1)}$ is tangent to B and verifies $\pounds_{\xi_Y^{(1)}|_B} \Pi = 0$.

*(2) A vector field ξ_Y on Y is said to be a **Noether symmetry**, or a **divergence symmetry** if*

$$\pounds_{\xi_Y^{(1)}|_B} \Theta_L = d\alpha$$

where α is the pullback to Z of a n-form on Y (that must be exact $\alpha = d\beta$ on B), $\xi_Y^{(1)}$ is tangent to B and verifies $\pounds_{\xi_Y^{(1)}{}_{|B}} \Pi = 0$.

(3) A vector field ξ_Z on Z is a ***Cartan symmetry*** *if*

$$\pounds_{\xi_Z}\Theta_L = d\alpha$$

where α is a n-form on Z (that is exact $\alpha = d\beta$ on B) (or, equivalently, if there is a n-form α' such that

$$\iota_{\xi_Z}\Omega_L = d\alpha'$$

we can put $\alpha' = \alpha + \iota_{\xi_Z}\Theta_L$), in other words, if ξ_Z is a Hamiltonian vector field), ξ_Z is tangent to B and verifies $\pounds_{\xi_{Z|B}}\Pi = 0$.

There is an obvious relation between these types of symmetries, completely analogous to those between the symmetries for the Euler-Lagrange equations, that is, a symmetry of the Lagrangian (resp. a Noether symmetry, Cartan symmetry) for the De Donder equations is a symmetry of the Lagrangian (resp. a Noether symmetry, Cartan symmetry) for the Euler-Lagrange equations.

Also note that a small computation shows that, in the case of of a Noether symmetry, α must be necessarily the pullback of a semibasic n-form on Y, locally expressed by

$$\alpha(x, y, z) = \alpha^{\mu}(x, y) d^n x_{\mu}$$

Note from the definition of Cartan symmetry that using Cartan's formula we obtain

$$\iota_{\xi_Z}\Omega_L = d(\iota_{\xi_Z}\Theta_L + \alpha)$$

and therefore $d\iota_{\xi_Z}\Omega_L = 0$, from where

$$\pounds_{\xi_Z}\Omega_L = 0$$

Theorem 5.14. ***(Noether's theorem)*** *If ξ_Z is a Cartan symmetry, such that $\pounds_{\xi_Z}\Theta_L = d\alpha$, then $\iota_{\xi_Z}\Theta_L - \alpha$ is a preserved quantity which is exact on the boundary.*

For the proof, repeat that of the Noether's theorem for Euler-Lagrange equations, where

$$\pounds_{\xi_Z}\Theta_L - d\alpha$$

now vanishes by definition.

In the case of a regular Lagrangian, and $n > 0$, a computation similar to that in Proposition 2.14 for the expression $\pounds_{\xi_Z}\Omega_L = 0$ produces two terms

$$\frac{\partial^2 L}{\partial z^i_{\mu}\partial z^j_{\nu}}\frac{\partial \xi^{\kappa}_X}{\partial y^k}\, dz^j_{\nu} \wedge dy^i \wedge dy^k \wedge d^{n-1}x_{\mu\kappa}$$

and

$$\frac{\partial^2 L}{\partial z^i_{\mu}\partial z^j_{\nu}}\frac{\partial \xi^{\kappa}_X}{\partial z^k_{\lambda}}\, dz^j_{\nu} \wedge dy^i \wedge dz^k_{\lambda} \wedge d^{n-1}x_{\mu\kappa},$$

which show that Cartan symmetries are automatically projectable. For this reason, and because projectable symmetries are typical of examples coming from Physics, we shall emphasize the role of vector fields which are projectable onto X.

Also note that the symmetries of Cartan preserve the horizontal subspaces for the connection formalism.

Proposition 5.15. *Assume that L is regular. If ξ_Z is a Cartan symmetry for the De Donder equations then ξ_Z preserves the horizontal distribution of any solution Γ satisfying* (6).

Proof. Since ξ_Z is a Cartan symmetry then $\pounds_{\xi_Z}\Omega_L = 0$. Therefore

$$\pounds_{\xi_Z} i_{\mathbf{h}}\Omega_L = 0$$

for any solution Γ of (6) with horizontal projector $\mathbf{h}$.

Hence,

$$\begin{aligned}
0 &= \left(\pounds_{\xi_Z} i_{\mathbf{h}}\Omega_L\right)(\xi_0, \xi_1, \ldots, \xi_n) \\
&= \xi_Z\left(i_{\mathbf{h}}\Omega_L(\xi_0, \xi_1, \ldots, \xi_n)\right) - \sum_{a=0}^{n} i_{\mathbf{h}}\Omega_L(\xi_1, \ldots, [\xi_Z, \xi_a], \ldots, \xi_n) \\
&= \sum_{b=0}^{n} \xi_Z\left(i_{\mathbf{h}(\xi_b)}\Omega_L(\xi_0, \ldots, \widehat{\xi_b}, \ldots, \xi_n)\right) \\
&\quad - \sum_{\substack{a, b = 0 \\ a \neq b}}^{n} (-1)^b i_{\mathbf{h}(\xi_b)}\Omega_L(\xi_0, \ldots, [\xi_Z, \xi_a], \ldots, \widehat{\xi_b}, \ldots, \xi_n) \\
&\quad - \sum_{b=0}^{n} (-1)^{b+1} i_{\mathbf{h}[\xi_Z, \xi_b]}\Omega_L(\xi_1, \ldots, \widehat{\xi_b}, \ldots, \xi_n) \\
&= \sum_{b=0}^{n}\left(\pounds_{\xi_Z} i_{\mathbf{h}(\xi_b)}\Omega_L\right)(\xi_0, \ldots, \widehat{\xi_b}, \ldots, \xi_n) - \sum_{b=0}^{n} i_{\mathbf{h}[\xi_Z, \xi_b]}\Omega_L(\xi_1, \ldots, \widehat{\xi_b}, \ldots, \xi_n)
\end{aligned}$$

First case ($n > 1$). Since Ω_L is multisymplectic and $\pounds_{\xi_Z}\Omega_L = 0$ we deduce that

$$[\xi_Z, \mathbf{h}(\xi)] = \mathbf{h}[\xi_Z, \xi] \qquad \forall \xi \in \mathfrak{X}(Z),$$

which implies that the horizontal distribution associated to Γ is $\mathbf{h}$-invariant

Second case ($n = 1$). Taking $\xi = \frac{\partial}{\partial t}$ then $\mathbf{h}(\xi) = \xi_L$ is the Reeb vector field of the cosymplectic structure (dt, Ω_L) (being L regular). Moreover, with the notation $d_t = \frac{d}{dt}$, we have

$$\mathbf{h}[\xi_Z, \frac{\partial}{\partial t}] = -d_t\tau\xi_L, \quad dt([\xi_Z, \xi_L)] = d_t\tau$$

where $dt(\xi_Z) = \tau$. Therefore,

$$dt([\xi_Z, \xi_L] - \mathbf{h}[\xi_Z, \frac{\partial}{\partial t}]) = 0$$

Since (Ω_L, dt) is a cosymplectic structure, we deduce that

$$[\xi_Z, \xi_L] = \mathbf{h}[\xi_Z, \frac{\partial}{\partial t}] = -d_t \tau \xi_L, \tag{17}$$

which implies the invariance of the distribution $\langle \xi_L \rangle$. Observe that equation (17) is the classical definition of dynamical symmetry for time-dependent mechanical systems.

Moreover, the boundary conditions are fulfilled since ξ_Z preserves B. ∎

Finally, we shall justify that these symmetries are really symmetries, in the sense that they transform solutions of the De Donder equations into new solutions of the De Donder equations.

Theorem 5.16. *The flow of Cartan symmetries maps solutions of the De Donder equations into solutions of the De Donder equations.*

Proof. If σ is a solution of the De Donder equation, and $\xi \in \mathfrak{X}(Z)$ is a Cartan symmetry having flow ϕ_t, and we define for each t

$$\psi_t := \pi_{XZ} \circ \phi_t \circ \sigma$$

then we claim that $\phi_t \circ \sigma \circ \psi_t^{-1}$ is a solution of the De Donder equations. Being the symmetry tangent to B, the boundary condition will be automatically satisfied.

As $\psi_0 = Id$, ψ_t is a local diffeomorphism for small $t's$. Therefore, $\phi_t \circ \sigma \circ \psi_t^{-1}$ makes sense for small $t's$. In order to prove

$$(\phi_t \circ \sigma \circ \psi_t^{-1})^*(\iota_X \Omega_L) = (\psi_t^{-1})^* \sigma^* \phi_t^*(\iota_X \Omega_L) = 0$$

it suffices to see that

$$\sigma^* \phi_t^*(\iota_X \Omega_L) = 0$$

for t in a neighbourhood of 0. Now for $t = 0$, this equation reduces to the De Donder equation, therefore, it suffices to see that

$$\sigma^*(\pounds_\xi \iota_X \Omega_L) = 0$$

Using again the De Donder equation,

$$0 = \sigma^*(\iota_{[\xi, X]} \Omega_L) = \sigma^*(\pounds_\xi \iota_X \Omega_L) - \sigma^*(\iota_X \pounds_\xi \Omega_L)$$

But

$$\pounds_\xi \Omega_L = -d\pounds_\xi \Theta_L = -dd\alpha = 0$$

which completes the proof. ∎

5.5 Symmetries for singular Lagrangian systems

For the singular Lagrangian case (described in section 2.7), we consider diffeomorphisms $\Psi : Z \to Z$ which preserve the Poincaré-Cartan $(n+2)$-form Ω_L (i.e. $\phi^*\Omega_L = \Omega_L$) and are π_{XZ}-projectable.0

Proposition 5.17. *If the diffeomorphism $\Psi : Z \longrightarrow Z$ verifying $\Psi(B) \subseteq B$ preserves the $(n+2)$-form Ω_L and it is π_{XZ}-projectable, then it restricts to a diffeomorphism $\Psi_a : Z_a \longrightarrow Z_a$, where Z_a is the a-ry constraint submanifold. Therefore, Ψ restricts to a diffeomorphism $\Psi_f : Z_f \longrightarrow Z_f$.*

Proof. If $z \in Z_1$ then there exists a linear mapping $\mathbf{h}_z : T_zZ \longrightarrow T_zZ$ such that $\mathbf{h}_z^2 = \mathbf{h}_z$, $\ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z$ and

$$i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z)$$

Consider the mapping

$$\mathbf{h}_{\Psi(z)} = T_z\Psi \circ \mathbf{h}_z \circ T_{\Psi(z)}\Psi^{-1}$$

It is clear that $\mathbf{h}_{\Psi(z)}$ is linear and $\mathbf{h}^2_{\Psi_z} = \mathbf{h}_{\Psi(z)}$ Moreover, since Ψ is π_{XZ} projectable then $\ker \mathbf{h}_{\Psi(z)} = (\mathcal{V}\pi_{XZ})_{\Psi(z)}$. Finally, since $\Psi^*\Omega_L = \Omega_L$ then

$$i_{\mathbf{h}_{\Psi(z)}}\Omega_L(\Psi(z)) = n\Omega_L(\Psi(z))$$

Therefore, if $z \in Z_1$ then $\Psi(z) \in Z_1$. Thus, the proposition is true if $a = 1$. Now, suppose that the proposition is true for $a = l$ and we shall prove that it is also true for $a = l+1$.

Let z be a point in Z_{l+1} then there exists $\mathbf{h}_z : T_zZ \longrightarrow T_zZ_l$ linear such that $\mathbf{h}_z^2 = \mathbf{h}_z$, $\ker \mathbf{h}_z = (\mathcal{V}\pi_{XZ})_z$ and $i_{\mathbf{h}_z}\Omega_L(z) = n\Omega_L(z)$. Since $\Psi(Z_l) \subseteq Z_l$ and Ψ is a diffeomorphism, then $T_z\Psi(T_zZ_l) \subseteq T_{\Psi(z)}Z_l$. Thus, $\mathbf{h}_{\Psi(z)} : T_{\Psi(z)}Z \longrightarrow T_{\Psi(z)}Z_l$ and $\Psi(z) \in Z_{l+1}$. We also have that $\mathbf{h}(TB_f) \subseteq TB_f$. ∎

Corollary 5.18. *Let ξ_Z be a π_{XZ}-projectable vector field on X such that $\pounds_{\xi_Z}\Omega_L = 0$, then ξ_Z is tangent to Z_f*

Corollary 5.19. *A Cartan symmetry which is π_{XZ}-projectable is tangent to Z_f*

Proposition 5.17 motivates the introduction of a more general class of symmetries. If Z_f is the final constraint submanifold and $i_{f1} : Z_f \longrightarrow Z$ is the canonical immersion then we may consider the $(n+2)$-form $\Omega_{Z_f} = i_{f1}^*\Omega_L$, the $(n+1)$-form $\Theta_{Z_f} = i_{f1}^*\Theta_L$ and now analyze a new kind of symmetries.

Definition 5.7. *A Cartan symmetry for the system (Z_f, Ω_{Z_f}) is a vector field on Z_f tangent to $Z_f \cap B$ such that $\pounds_{\xi_{Z_f}}\Theta_{Z_f} = d\alpha_{Z_f}$, for some $\alpha_{Z_f} \in \Lambda^n Z_f$.*

If it is clear that if ξ_Z is a Cartan symmetry of the De Donder equations then using Proposition 5.17 we deduce that $X_{|Z_f}$ is a Cartan symmetry for the system (Z_f, Ω_{Z_f}).

5.6 Symmetries in the Hamiltonian formalism

We can define as well symmetries in the Hamiltonian formalism as we did for the De Donder equation, which are closely related by the equivalence theorem.

Definition 5.8. *Given a Hamiltonian h, we have the following definitions of symmetries for the Hamilton equations:*

(1) A vector field ξ_Y on Y is said to be a ***Noether symmetry****, or a* ***divergence symmetry*** *if there exists a semibasic n-form on Y whose pullback α to $\Lambda_2^{n+1}Y$ (which is exact $\alpha = d\beta$ on B^*) and verifies*

(a) The α-lift of ξ_Y to $\Lambda_2^{n+1}Y$ is projectable to a vector field $\xi_Y^{(1*)}$

(b) $\pounds_{\xi_Y^{(1*)}}\Theta_h = d\alpha$, $\xi_Y^{(1*)}$ is also tangent to B^* and verifies $\pounds_{\xi_Y^{(1*)}|_{B^*}}\pi_{XZ^*} = 0$.

(2) A vector field ξ_Z on Z^* is a ***Cartan symmetry*** *if*

$$\pounds_{\xi_Z}\Theta_h = d\alpha$$

where α is an n-form on Z^* (which is exact $\alpha = d\beta$ on B^*), ξ_Z is also tangent to B^* and verifies $\pounds_{\xi_Z|_{B^*}}\pi_{XZ^*} = 0$

As usual, Noether symmetries induce Cartan symmetries on Z^*.

Supose that ξ is a vector field on Y, and α is the pull-back to $\Lambda_2^{n+1}Y$ of a π_{XY}-semibasic form on Y. If the α-lift of ξ to $\Lambda_2^{n+1}Y$ projects onto a vector field on Z^* then ξ_Y is a Noether symmetry.

Theorem 5.20. ***(Noether's theorem)*** *If ξ_{Z^*} is a Cartan symmetry, such that $\pounds_{\xi_{Z^*}}\Theta_h = d\alpha$, then $\sigma^* d(\iota_{\xi_{Z^*}}\Theta_h - \alpha) = 0$ for every solution σ of the Hamilton equations. Furthermore, $\iota_{\xi_{Z^*}}\Theta_h - \alpha$ is exact on ∂Z^*.*

This theorem is entirely analogous to that of the Noether's theorem for De Donder equations.

Finally, we shall justify that these are real symmetries, in the sense that they transform solutions of the Hamilton equations into new solutions of the Hamilton equations.

Theorem 5.21. *The flow of Cartan symmetries maps solutions of the Hamilton equations into solutions of the Hamilton equations.*

The proof is identical to that given for the De Donder equations in theorem 5.16.

5.7 The Legendre transformation and the symmetries

In this section we shall finally relate the symmetries of the De Donder equations to the symmetries of the Hamiltonian formalism, under the assumption of hyperregularity. Within this section, we shall assume that L is a hyperregular Lagrangian.

Proposition 5.22. *If ξ_Z is a Cartan symmetry for the De Donder equation, then $Tleg_L(\xi_Z)$ is a Cartan symmetry for the Hamilton equations. The converse is also true.*

Proof. If we just apply $(leg_L^{-1})^*$ to the Cartan condition for the De Donder equations we get the Cartan condition for the Hamilton equations:

$$0 = (leg_L^{-1})^*(\pounds_{\xi_Z}\Theta_L - d\alpha) = \pounds_{Tleg_L(\xi_Z)}(leg_L^{-1})^*\Theta_L - d\tilde{\alpha} = \pounds_{Tleg_L(\xi_Z)}\Theta_h - d\tilde{\alpha}.$$

where $leg_L^*\tilde{\alpha} = \alpha$. Boundary preservation is trivial, because of the way B^* has been defined, and the compatibility with the Legendre map. ∎

In a similar way we prove the following result

Lemma 5.23. *If ξ_Y is a Noether symmetry for the De Donder equation, such that $\pounds_{\xi_Y^{(1)}}\Theta_L - d\alpha$, then $TLeg_L(\xi_Y^{(1)})$ is the α-lift of ξ_Y.*

From which we can obtain

Proposition 5.24. *Every Noether symmetry for the De Donder equations is a Noether symmetry for the Hamilton equations. The converse is also true.*

Proof. We have that

$$Tleg_L(\xi_Y^{(1)}) = (T\mu \circ TLeg_L)(\xi_Y^{(1)})$$

therefore the α-lift of ξ_Y projects onto $Tleg_L(\xi_Y^{(1)})$ on Z^*, and as $\xi_Y^{(1)}$ is a Cartan symmetry, its image $Tleg_L(\xi_Y^{(1)})$ also verifies the Cartan condition (as $\pounds_{Tleg_L(\xi_Y^{(1)})}\Theta_h - d\tilde{\alpha} = \pounds_{Tleg_L(\xi_Y^{(1)})}(leg_L^{-1})^*\Theta_L - d(leg_L^{-1})^*\alpha = (leg_L^{-1})^*(\pounds_{\xi_Y^{(1)}}\Theta_L - d\alpha) = 0$). As usual, boundary conditions are trivially fulfilled. ∎

5.8 Symmetries in the Hamiltonian formalism for almost regular Lagrangians

On the final constraint submanifold M_f we have the following definition.

Definition 5.9. *A Cartan symmetry for the system (M_f, Ω_{M_f}) is a vector field on M_f tangent to $M_f \cap B^*$ such that $\pounds_{\xi_{M_f}}\Theta_{M_f} = d\alpha_{M_f}$, for some $\alpha_{M_f} \in \Lambda^n M_f$.*

Proposition 5.25. *If ξ_{M_f} is a Cartan symmetry of (M_f, Ω_{M_f}) then any vector field ξ_{Z_f}, such that $Tleg_f(\xi_{Z_f}) = \xi_{M_f}$ is a Cartan symmetry of (Z_f, Ω_{Z_f}).*

5.9 Symmetries on the Cauchy data space

The symmetries of presymplectic systems were exhaustively studied by two of the authors in [50, 51] (see also [14, 32]). In [50] (Proposition 4.1 and Corollary 4.1) it was proved that for a general presymplectic system given by (M, ω, Λ), where M is a differentiable manifold, ω a closed 2-form and Λ a closed 1-form, a vector field ξ such that

$$i_\xi \omega = dG,$$

where $G : M \to \mathbb{R}$, is a Cartan symmetry of the presymplectic system (for $\Lambda = 0$). In fact, given a solution U for the presymplectic system, since U satisfies $\iota_U \omega = 0$, then we have

$$0 = \iota_U \iota_\xi \omega = U(G).$$

The following proposition explains the relationship between Cartan symmetries of the De Donder equations and Cartan symmetries for the presymplectic system $(\tilde{Z}, \widetilde{\Omega})$.

Proposition 5.26. *Let ξ_Z be a Cartan symetry of the De Donder equations, that is, $\pounds_{\xi_Z} \Theta_L = d\alpha$. Then the induced vector field $\xi_{\tilde{Z}}$ in $\tilde{Z}$, defined by $\xi_{\tilde{Z}}(\gamma) = \xi_Z \circ \gamma$, is a Cartan symmetry of the presymplectic system $(\tilde{Z}, \widetilde{\Omega_L})$.*

Proof: If $\pounds_{\xi_Z} \Theta_L = d\alpha$, then

$$i_{\xi_Z} \Omega_L = d(\alpha - i_{\xi_Z} \Theta_L)$$

that is, ξ_Z is a Hamiltonian vector field for the n form $\beta = \alpha - i_{\xi_Z} \Theta_L$. Then from Proposition 4.8 we have

$$i_{\widetilde{\xi_Z}} \widetilde{\Omega_L} = d\tilde{\beta}$$

which shows that $\widetilde{\xi_Z}$ is a Cartan symmetry for the presymplectic system $(\tilde{Z}, \widetilde{\Omega_L})$. ∎

5.10 Conservation of preserved quantities along solutions

Proposition 5.27. *If α is a preserved quantity, and $c_{\tilde{Z}}$ is a solution of the De Donder equations* (12) *such that its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X and α is exact on $B \subseteq \partial Z$ ($\alpha_{|B} = d\beta$), then $\tilde{\alpha} \circ c_{\tilde{Z}}$ is constant; in other words, the following function*

$$\int_M c_{\tilde{Z}}(t)^* \alpha - \int_{\partial M} c_{\tilde{Z}}(t)^* \beta$$

is constant with respect to t.

Proof. Pick $t_1 < t_2$ two real numbers in the domain of the solution curve, and let us denote by $M_1 = c_{\tilde{X}}(t_1)$ and $M_2 = c_{\tilde{X}}(t_2)$. As $c_{\tilde{X}}$ splits X, then we can consider the piece $U \subseteq X$

identified with $M \times [t_1, t_2]$, M_1 is identified with $M \times t_1$, M_2 is identified with $M \times t_2$, and let us denote by V the boundary piece corresponding to $\partial M \times [t_1, t_2]$. On view of (11), then

$$c_{\tilde{Z}}(t)^* d\alpha = 0 \quad \text{for all} \quad t$$

whence if we integrate and apply Stoke's theorem, we get

$$0 = \int_{M_2} c_{\tilde{Z}}(t)^* \alpha + \int_V c_{\tilde{Z}}(t)^* \alpha - \int_{M_1} c_{\tilde{Z}}(t)^* \alpha$$

If we put $\alpha = d\beta$ on B, then $0 = \partial\partial U = \partial M_2 + \partial V - \partial M_1$, whence applying Stoke's theorem again, we obtain

$$\int_V c_{\tilde{Z}}(t)^* \alpha = \int_{\partial V} c_{\tilde{Z}}(t)^* \beta = \int_{\partial M_1} c_{\tilde{Z}}(t)^* \beta - \int_{\partial M_2} c_{\tilde{Z}}(t)^* \beta.$$

∎

Corollary 5.28. *In particular, if ξ_Y is a symmetry of the Lagrangian for the De Donder equations , then the preceding formula can be applied to the preserved quantity $\iota_{\xi_Y^{(1)}} \Theta_L$ and we get that the following integral is preserved along solutions of the De Donder equations* (12) *such that its projection $c_{\tilde{X}}$ to $\tilde{X}$ splits X*

$$\int_M c_{\tilde{Z}}(t)^* \iota_{\xi_Y^{(1)}} \Theta_L + \int_{\partial M} c_{\tilde{Z}}(t)^* \iota_{\xi_Y^{(1)}} \Pi$$

The preceding formula can also be found on [3].

5.11 Localizable symmetries. Second Noether's theorem

Definition 5.10. *A symmetry of the lagrangian ξ_Y is said to be **localizable** when $\xi_Y^{(1)}$ it vanishes on ∂Z and for every pair of open sets U and U' in X with disjoint closures, there exists another symmetry of the lagrangian ζ_Y such that*

$$\xi_Y^{(1)} = \zeta_Y^{(1)} \qquad \textit{on } \pi_{XZ}^{-1}(U)$$

and

$$\zeta_Y^{(1)} = 0 \qquad \textit{on } \pi_{XZ}^{-1}(U') \cup \partial Z$$

Theorem 5.29. ***Second Noether Theorem.*** *If ξ_Y is a localizable symmetry, and $c_{\tilde{Z}}$ is a solution of De Donder equations* (12)*, then*

$$(\widetilde{\iota_{\xi_Y} \Theta_L})(c_{\tilde{Z}}(t)) = 0$$

for all t. Therefore, if $\alpha = \iota_\xi \Theta_L$ is the preserved quantity, then $\tilde{\alpha}$ is a constant of motion for the De Donder equations.

Proof. First Noether theorem guarantees that the preceding application is constant. Pick t_0 in the domain of definition of $c_{\tilde{Z}}$, the space-time decomposition of X guarantees that, for $t \neq t_0$, we can find, using tubular neighbourhoods, two disjoint open sets U and U' with disjoint closures containing $Im(c_{\tilde{Z}}(t_0))$ and $Im(c_{\tilde{Z}}(t))$ respectively.

If ζ_Y is the Cartan symmetry whose existence guarantees the notion of localizable symmetry, respect to U and U', then

$$(\widetilde{\iota_{\xi_Y}\Theta_L})(c_{\tilde{Z}}(t_0)) = (\widetilde{\iota_{\zeta_Y}\Theta_L})(c_{\tilde{Z}}(t_0)) = (\widetilde{\iota_{\zeta_Y}\Theta_L})(c_{\tilde{Z}}(t)) = 0.$$

∎

6 Momentum map

In this section we are interested in considering groups of symmetries acting on the configuration space Y, which induce a lifted action into Z which preserves the Lagrangian form.

6.1 Action of a group

If G is a Lie group acting on Y, then the action of G on Y can be lifted to an action of G on Z, and the infinitesimal generator of the lifted action corresponds to the lift of the infinitesimal generator of the action, in other words,

$$\xi_Z = \xi_Y^{(1)}$$

Definition 6.1. *We shall say that a Lie group G acts as a* ***group of symmetries of the Lagrangian*** *if it defines an action on Y that projects onto a compatible action on X, which 1-jet prolongation preserves B, and if the flow ϕ_Z of ξ_Z verifies*

$$\phi_Z^*\mathcal{L} = \mathcal{L} \qquad \phi_Z^*\Pi = \Pi$$

The fact that the action is fibred implies that ξ_Y is a projectable vector field. Therefore, the condition $\phi_Z^*\mathcal{L} = \mathcal{L}$, infinitesimally expressed as

$$\pounds_{\xi_Z}\mathcal{L} = 0,$$

jointly with the following two direct consequences of the definition:

(i) ξ_Z is tangent to B

(ii) $\pounds_{(\xi_Z)_{|B}}\Pi = 0$,

states the fact that ξ_Y is a symmetry of the Lagrangian.

6.2 Momentum map

If we have a group of symmetries of the Lagrangian G acting on Y, we can make use of the Poincaré-Cartan $(n+1)$-form on Z to construct the analogous of the momentum map in Classical Mechanics.

Definition 6.2. *The* ***momentum map*** *is a mapping*

$$J : Z \longrightarrow \mathfrak{g}^* \otimes \Lambda^n Z$$

or alternatively,

$$J : Z \otimes \mathfrak{g} \longrightarrow \Lambda^n Z$$

defined by $J(z,\xi) := (\iota_{\xi_Z}\Theta_L)_z$.

Therefore, $J(\cdot,\xi)$ *is a* n*-form, that we shall denote by* J^ξ.

Remark 6.1. *On* B*, since* $\pounds_{(\xi_Z)_{|B}}\Pi = 0$ *we have that* $\iota_{(\xi_Z)_{|B}} d\Pi = -d\iota_{(\xi_Z)_{|B}}\Pi$*, and therefore,*

$$J(z,\xi) = (\iota_{\xi_Z}\Theta_L|_B)(z) = (\iota_{\xi_Z} d\Pi)(z) = -(d\iota_{\xi_Z}\Pi)(z)$$

Notice that J^ξ is a preserved quantity, and we called $\widetilde{J^\xi}$ its associated momentum.

Proposition 6.2.

$$dJ^\xi = \iota_{\xi_Z}\Omega_L$$

Proof. As ξ is projectable, $\pounds_{\xi_Z}\Theta_L = 0$ (by 5.3), whence

$$0 = \pounds_{\xi_Z}\Theta_L = \iota_{\xi_Z} d\Theta_L + d\iota_{\xi_Z}\Theta_L = -\iota_{\xi_Z}\Omega_L + dJ^\xi.$$

∎

6.3 Momentum map in Cauchy data spaces

If G is a Lie group acting on Y as symmetries of the Lagrangian, it induces an action on $\tilde{Z}$ defined pointwise on the image of every curve in $\tilde{Z}$.

For $\xi \in \mathfrak{g}$, the vector field $\xi_{\tilde{Z}}$ is precisely the vector on $\tilde{Z}$ induced by the vector field ξ_Z on Z. And since ξ_Z is a Cartan symmetry, so is $\xi_{\tilde{Z}}$.

In a similar manner, the presymplectic form $\widetilde{\Theta_L}$ induces a momentum map

$$\tilde{J} : \tilde{Z} \longrightarrow \mathfrak{g}^*$$

defined using its pairing (for $\xi \in \mathfrak{g}$)

$$\tilde{J}^\xi = \langle \tilde{J}, \xi \rangle : \tilde{Z} \longrightarrow \mathbb{R}$$

by

$$\tilde{J}^\xi := \iota_{\xi_{\tilde{Z}}}\widetilde{\Theta_L}$$

One immediately has that $\widetilde{J^\xi} = \tilde{J}^\xi$. As we know that a Cartan symmetry for the De Donder equations in Z, then $\tilde{\xi}$ is a Cartan symmetry for the De Donder equations in $\tilde{Z}$, thus $\tilde{J}^\xi$ is a preserved quantity for the presymplectic setting.

By repeating the arguments in (6.2), we have:

Proposition 6.3.

$$d\tilde{J}^\xi = \iota_{\xi_{\tilde{Z}}}\widetilde{\Omega_L}$$

7 Examples

7.1 The Bosonic string

Let X be a 2-dimensional manifold, and (B, g) a $(d+1)$-dimensional spacetime manifold endowed with a Lorentz metric g of signature $(-, +, \ldots, +)$. A *bosonic string* is a map $\phi : X \longrightarrow B$ (see [1, 28]).

In the folllowing, we shall follow the Polyakov approach to clasical bosonic string theory. Let $S_2^{1,1}(X)$ be the bundle over X of symmetric covariant rank two tensors of Lorentz signature $(-, +)$ or $(1, 1)$. We take the vector bundle $\pi : Y = X \times B \times S_2^{1,1}(X) \longrightarrow X$. Therefore, in this formulation, a field ψ is a section (ϕ, s) of the vector bundle $Y = X \times B \times S_2^{1,1}(X) \longrightarrow X$, where $\phi : X \longrightarrow X \times B$ is the bosonic string and s is a Lorentz metric on X.

7.1.1 Lagrangian description

We have that $Z = J^1(X \times B) \times_X J^1(S_2^{1,1}(X))$. Taking coordinates (x^μ), (y^i) and $(x^\mu, s_{\mu\zeta})$ on X, B and $S_2^{1,1}(X)$ then the canonical local coordinates on Z are $(x^\mu, y^i, s_{\zeta\xi}, y^i_\mu, s_{\zeta\xi\mu})$. In this system of local coordinates, the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^i_\zeta y^j_\xi d^2x \ .$$

The Cartan 2-form is

$$\Theta_L = \sqrt{-\det(s)}\left(-s^{\mu\nu}g_{ij}y^j_\nu dy^i \wedge d^1x_\mu + \frac{1}{2}s^{\mu\nu}g_{ij}y^i_\mu y^j_\nu d^2x\right)$$

and the Cartan 3-form is

$$
\begin{aligned}
\Omega_L &= dy^i \wedge d\left(-\sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^j_\xi\right) \wedge d^1x_\zeta \\
&\quad -d\left(\frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^i_\zeta y^j_\xi\right) \wedge d^2x \\
&= -\frac{1}{2}\left(\frac{\partial\sqrt{-\det(s)}}{\partial s_{\rho\sigma}}s^{\zeta\xi}g_{ij}y^i_\zeta y^j_\xi - \sqrt{-\det(s)}s^{\zeta\rho}s^{\xi\sigma}g_{ij}y^i_\eta y^j_\xi\right) ds_{\rho\sigma} \wedge d^2x \\
&\quad -\frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{ij}}{\partial y^k}y^i_\zeta y^j_\xi \, dy^k \wedge d^2x - \sqrt{-\det(s)}s^{\zeta\xi}g_{ij}y^i_\zeta \, dy^j_\xi \wedge d^2x \\
&\quad +\left(\frac{\partial\sqrt{-\det(s)}}{\partial h_{\rho\sigma}}s^{\zeta\xi}g_{ij}y^j_\xi - \sqrt{-\det(s)}s^{\zeta\rho}s^{\xi\sigma}g_{ij}y^j_\xi\right) ds_{\rho\sigma} \wedge dy^i \wedge d^1x_\zeta \\
&\quad +\sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{ij}}{\partial y^k}y^j_\xi \, dy^k \wedge dy^i \wedge d^1x_\zeta \\
&\quad +\sqrt{-\det(s)}s^{\zeta\xi}g_{ij} \, dy^j_\xi \wedge dy^i \wedge d^1x_\zeta .
\end{aligned}
$$

If we solve the equation $i_{\mathbf{h}}\Omega_L = \Omega_L$, where

$$
\mathbf{h} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^i_\mu\frac{\partial}{\partial y^i} + \gamma_{\zeta\xi\mu}\frac{\partial}{\partial s_{\zeta\xi}} + \Gamma^i_{\zeta\mu}\frac{\partial}{\partial y^i_\zeta} + \gamma_{\zeta\xi\rho\mu}\frac{\partial}{\partial s_{\zeta\xi\rho}}\right) ,
$$

we obtain that:

$$
\begin{aligned}
\Gamma^i_\mu &= y^i_\mu \\
0 &= \frac{1}{2}\sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{ij}}{\partial y^k}y^i_\zeta y^j_\xi - \sqrt{-\det(s)}s^{\zeta\xi}\frac{\partial g_{kj}}{\partial y^i}y^i_\zeta y^j_\xi - \sqrt{-\det(s)}s^{\zeta\xi}g_{kj}\Gamma^j_{\xi\zeta} \\
&\quad -\left(\frac{\partial\sqrt{-\det(s)}}{\partial s_{\rho\sigma}}s^{\zeta\xi}g_{kj}y^j_\xi - \sqrt{-\det(s)}s^{\zeta\rho}s^{\xi\sigma}g_{kj}y^j_\xi\right)\gamma_{\rho\sigma\zeta} ,
\end{aligned}
$$

and the constraints given by the equations

$$
\frac{\partial}{\partial s_{\rho\theta}}\left(\sqrt{-\det(s)}s^{\zeta\xi}\right) g_{ij}y^i_\zeta y^j_\xi = 0 .
$$

The previous equation corresponds to the three following constraints

$$
\begin{aligned}
\left[s^{\zeta 0}s^{\xi 0}(s_{01}^2 - s_{00}s_{11}) + \frac{1}{2}s^{\zeta\xi}s_{11}\right] g_{ij}y^i_\zeta y^j_\xi &= 0 \\
\left[s^{\zeta 1}s^{\xi 1}(s_{01}^2 - s_{00}s_{11}) + \frac{1}{2}s^{\zeta\xi}s_{00}\right] g_{ij}y^i_\zeta y^j_\xi &= 0 \\
\left[s^{\zeta 0}s^{\xi 1}(s_{01}^2 - s_{00}s_{11}) - s^{\zeta\xi}s_{01}\right] g_{ij}y^i_\zeta y^j_\xi &= 0
\end{aligned}
$$

which determine Z_2.

7.1.2 Hamiltonian description

The Legendre transformation is given by

$$Leg_L(x^\mu, y^i, s_{\zeta\xi}, y^i_\mu, s_{\zeta\xi\mu}) = (x^\mu, y^i, s_{\zeta\xi}, -\sqrt{-\det(s)}\, s^{\mu\zeta} g_{ij} y^j_\zeta, 0)$$

Therefore, the Lagrangian L is almost-regular and, moreover, $\tilde{M}_1 = \mathrm{Im}\, Leg_L \cong M_1 = leg_L(Z) \cong J^1(X \times B) \times_X S^{1,1}_2(X)$. Take now coordinates $(x^\mu, y^i, s_{\zeta\xi}, p^\mu_i)$ on M_1 and consider the mapping $s_1 : M_1 \to \tilde{M}_1$ given by

$$s_1(x^\mu, y^i, s_{\zeta\xi}, p^\mu_i) = (x^\mu, y^i, s_{\zeta\xi}, p = \frac{1}{2\sqrt{-\det(s)}} s_{\zeta\xi} g^{ij} p^i_\zeta p^j_\xi, p^\mu_i)$$

Then, we have

$$\Omega_{M_1} = -d\left(\frac{1}{2\sqrt{-\det(s)}} s_{\zeta\xi} g^{ij} p^\zeta_i p^\xi_j\right) \wedge d^2x + dy^i \wedge dp^\mu_i \wedge d^1x_\mu$$

and the Hamilton equations are given by $i_{\tilde{\mathbf{h}}}\Omega_{M_1} = \Omega_{M_1}$. Putting

$$\tilde{\mathbf{h}} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \tilde{\Gamma}^i_\mu \frac{\partial}{\partial y^i} + \tilde{\gamma}_{\zeta\xi\mu} \frac{\partial}{\partial s_{\zeta\xi}} + \tilde{\Gamma}^\zeta_{i\mu} \frac{\partial}{\partial p^\zeta_i}\right)$$

we obtain

$$\begin{aligned}
\tilde{\Gamma}^i_\mu &= -\frac{1}{\sqrt{-\det(s)}} s_{\zeta\mu} g^{ij} p^\zeta_j \\
\tilde{\Gamma}^\mu_{i\mu} &= \frac{1}{2\sqrt{-\det(s)}} s_{\zeta\xi} \frac{\partial g^{ij}}{\partial y^k} p^i_\zeta p^j_\xi \ ,
\end{aligned}$$

and the secondary constraints

$$\frac{g^{ij}}{\sqrt{-\det(s)}} \left(\frac{1}{2\det(s)} \frac{\partial \det(s)}{\partial s_{\rho\sigma}} s_{\zeta\xi} p^\zeta_i p^\xi_j - p^\rho_i p^\sigma_j\right) = 0$$

determining M_2.

7.1.3 Symmetries

Let λ be an arbitrary function on X, and we denote also by λ its pullback to Y and Z.

Consider the following $\pi_{XY}-$projectable vector field on Y

$$\xi_Y := \lambda s_{\sigma\rho} \frac{\partial}{\partial s_{\sigma\rho}}$$

Its 1-jet prolongation is given by

$$\xi_Z := \xi_Y^{(1)} = \lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}} + \left(\frac{\partial \lambda}{\partial x^\mu} s_{\sigma\rho} + \lambda s_{\sigma\rho,\mu}\right)\frac{\partial}{\partial s_{\sigma\rho,\mu}}$$

We shall prove that ξ_Y is a symmetry of the Lagrangian. Note that

$$\begin{aligned}\pounds_{\xi_Z}\Theta_L = \pounds_{\xi_Y}(\sqrt{-\det(s)})\left(-s^{\mu\nu}g_{ij}y^j_\nu dy^i \wedge d^1x_\mu + \frac{1}{2}s^{\mu\nu}g_{ij}y^i_\mu y^j_\nu d^2x\right)\\ + \sqrt{-\det(s)}\left(-\pounds_{\xi_Y}(s^{\mu\nu})g_{ij}y^j_\nu dy^i \wedge d^1x_\mu + \frac{1}{2}\pounds_{\xi_Y}(s^{\mu\nu})g_{ij}y^i_\mu y^j_\nu d^2x\right)\end{aligned}$$

And a little computation shows that

$$\xi_Y(\sqrt{-\det(s)}) = \lambda\sqrt{-\det(s)}$$

and

$$\pounds_{\xi_Y}(s^{\mu\nu}) = -\lambda s^{\mu\nu}$$

Therefore, ξ_Y is a symmetry of the Lagrangian, and as the corresponding Cartan symmetry ξ_Z is π_{XZ} projectable, then the symmetry projects onto the final constraint manifold.

The preserved quantity given by Noether's theorem is given by

$$J^{\xi_Y} = \sum_{\sigma,\rho,\mu} \lambda s_{\sigma\rho,\mu}s_{\sigma\rho}d^1x_\mu$$

Note that the vector field

$$\xi_Y = 2\lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}}$$

is the infinitesimal generator of the action of the group $N = \mathcal{C}S^{1,1}_2(X) \equiv \mathcal{F}(X, \mathbb{R}^+)$ of the conformal transformations of a metric of signature $(1,1)$ given by

$$\lambda(\phi, s) := (\phi, \lambda^2 s)$$

We have that

$$det(\lambda^2 s) = \lambda^4 det(s)$$

and

$$(\lambda^2 s)^{\mu\nu} = \lambda^{-2}s^{\mu\nu};$$

therefore, the action preserves the constraint equations.

In a similar manner, we can consider the action of $H = Diff(X)$ by

$$\eta(\phi, s) := (\phi \circ \eta^{-1}, (\eta^{-1})^* s)$$

or more generally, consider the semidirect product $G = H[N]$, where the action of elements $\eta \in H$ on elements $\lambda \in N$ is given by

$$\eta \cdot \lambda := \lambda \circ \eta^{-1}$$

The group G is a group of symmetries for Y, and the action is given by

$$(\eta, \lambda) \cdot (\phi, s) := (\phi \circ \eta^{-1}, \lambda^2 (\eta^{-1})^* s)$$

7.1.4 Symmetries on the Hamiltonian side

Not being L regular, we cannot guarantee that ξ_Y is a symmetry of the Lagrangian for the Hamiltonian side. However, an easy computation gives us that

$$\xi_Y^{(1)} = \lambda s_{\sigma\rho} \frac{\partial}{\partial s_{\sigma\rho}} - \lambda p^{\mu}_{\sigma\rho} \frac{\partial}{\partial p^{\mu}_{\sigma\rho}}$$

Thus,

$$\pounds_{\xi_Y^{(1)}} \Theta_L = \pounds_{\xi_Y^{(1)}} (p^{\mu}_{\sigma\rho} d s_{\sigma\rho} d^n x_{\mu}) = p^{\mu}_{\sigma\rho} s_{\sigma\rho} \frac{\partial \lambda}{\partial x^{\mu}} d^2 x$$

However, note that in M_1 we have that $p^{\mu}_{\sigma\rho} = 0$, therefore ξ_Y restricts to a symmetry there of the form

$$\lambda s_{\sigma\rho} \frac{\partial}{\partial s_{\sigma\rho}}$$

Furthermore, this is the infinitesimal generator of the restriction of the lifted action on Z^*, and one easily deduces, on view of the form of the secondary constrain equation, that the action restricts as well to the secondary constraint submanifold.

7.1.5 More symmetries

In general, one can consider the invariance of the equations and the Lagrangian respect to diffeomorphisms of X. If η is one of such diffeomorphisms, then $\eta(\phi, s) = (\phi \circ \eta^{-1}, (\eta^{-1})^* s)$, having infinitesimal generator

$$-(s_{\sigma\mu} \frac{\partial \xi^{\mu}}{\partial x^{\rho}} + s_{\rho\mu} \frac{\partial \xi^{\mu}}{\partial x^{\sigma}}) \frac{\partial}{\partial s_{\sigma\rho}} + \xi^{\mu} \frac{\partial}{\partial x^{\mu}}$$

where $\xi^{\mu} \frac{\partial}{\partial x^{\mu}}$ is the infinitesimal generator of η.

The most general situation arises when considering the semidirect product $H[N]$ of the group $H = Diff(X)$ and the group N of the positive real functions on X defined above, given by

$$\eta \cdot \lambda := \lambda \circ \eta^{-1}$$

The action is defined as follows

$$(\eta, \lambda)(\phi, s) = (\phi \circ \eta^{-1}, \lambda^2(\eta^{-1})^* s),$$

and the infinitesimal generator is

$$2\lambda s_{\sigma\rho}\frac{\partial}{\partial s_{\sigma\rho}} - (s_{\sigma\mu}\frac{\partial \xi^\mu}{\partial x^\rho} + s_{\rho\mu}\frac{\partial \xi^\mu}{\partial x^\sigma})\frac{\partial}{\partial s_{\sigma\rho}} + \xi^\mu \frac{\partial}{\partial x^\mu}$$

This is proved to be a symmetry of the Lagrangian (see [28]), and the corresponding preserved quantity is

$$\frac{\partial L}{\partial y^i}(y^i_\mu \xi^\nu) + \frac{\partial L}{\partial s_{\sigma\rho}}(s_{\sigma\rho,\nu}\xi^\nu - 2\lambda s_{\sigma\rho} + s_{\sigma\nu}\frac{\partial \xi^\nu}{\partial x^\rho} + s_{\rho\nu}\frac{\partial \xi^\nu}{\partial x^\sigma}) = 0$$

for $\lambda, \xi^\nu and \frac{\partial \xi^\nu}{\partial x^\rho}$ arbitrary, which gives in particular the equation $\partial L/\partial s_{\sigma\rho} = 0$, which is expanded into

$$\frac{1}{2}s^{\mu\nu}g_{ij}y^i_\mu y^j_\nu s_{\sigma\rho} = g_{ij}y^i_\sigma y^j_\rho$$

which amounts to say that h is a metric conformally equivalent to $\phi^* g$ and that the conformal factor is precisely $\frac{1}{2}s^{\mu\nu}g_{ij}y^i_\mu y^j_\nu$.

7.2 Klein-Gordon equations

7.2.1 Lagrangian setting

For the Klein-Gordon equation, we set (X, g) be a Minkovski space, and $Y := X \times \mathbb{R}$, where $\pi : Y \longrightarrow X$ is the first canonical projection. A section ϕ of π can be identified with a smooth function on X, say $\varphi \in \mathcal{C}^\infty(X)$, where $y(j^1\phi(x)) = \varphi(x)$ and $z_\mu(j^1\phi(x)) = \dfrac{\partial \varphi}{\partial x^\mu}(x)$.

The chosen volume form will be $\eta := \sqrt{-det\, g}$.

7.2.2 Lagrangian setting

The Lagrangian function will be

$$L(x^\mu, y, z_\mu) := \frac{1}{2}\left(g^{\mu\nu}z_\mu z_\nu + m^2 y^2\right)$$

which is regular, as

$$\hat{p}^\mu = \frac{\partial L}{\partial z_\mu} = g^{\mu\nu}z_\nu$$

and thus the Hessian matrix is precisely $(g^{\mu\nu})$.

The Poincaré-Cartan 4-form is

$$\Theta_L = \sqrt{-det\, g}\left(g^{\mu\nu}z_\mu dy \wedge d^3x_\mu - \frac{1}{2}(g^{\mu\nu}z_\mu z_\nu - m^2y^2)d^4x\right)$$

The boundary condition will be $B = 0$, that is, $\sigma(\partial X) = 0$, and this restriction is required as an asymptotic condition to replace the restrictions of compactness that we have placed on X.

And the Euler-Lagrange equations in terms of φ become

$$m^2\varphi = g^{\mu\nu}\frac{\partial^2\varphi}{\partial x^\mu \partial x^\nu}$$

that is, the Klein-Gordon equation.

7.2.3 Legendre transformation and Hamiltonian setting

We compute

$$\hat{p} = \frac{1}{2}(-g^{\mu\nu}z_\mu z_\nu + m^2y^2)\sqrt{-det\, g}$$

Thus we can write the Hamiltonian

$$H(x^\mu, y, p^\mu) = \frac{1}{2}(g_{\mu\nu}p^\mu p^\nu + m^2y^2),$$

and the Hamilton equation for φ corresponding to a section $\phi(x^\mu) = (x^\mu, \varphi(x^\mu), \varphi^\mu(x^\mu))$ become

$$\frac{\partial\varphi}{\partial x^\mu} = g_{\mu\nu}p^\nu$$
$$\sum_\mu \frac{\partial\varphi^\mu}{\partial x^\mu} = (\sqrt{-detg})m^2\varphi$$

7.2.4 Symmetries

Let ξ_X be a Killing vector field on X, with coordinates

$$\xi_X = \xi_\mu\frac{\partial}{\partial x^\mu}$$

Let us call ξ_Y the vector field ξ_X as seen in Y, that is, locally,

$$\xi_Y(x,t) := \xi_\mu\frac{\partial}{\partial x^\mu}$$

Its 1-jet prolongation ξ_Z is given by

$$\xi_Z = \xi_\mu\frac{\partial}{\partial x^\mu} - z_\nu\frac{d\xi^\nu}{dx^\mu}\frac{\partial}{\partial z_\mu}$$

These vector fields are symmetries of the Lagrangian, and the associated preserved quantity is written as

$$\left[-g^{\mu\nu}z_\mu\xi^\gamma dy \wedge d^2x_{\nu\gamma} - \frac{\xi^\nu}{2}\left(g^{\mu\nu}z_\mu z_\nu - m^2y^2\right)d^3x_\gamma\right]\sqrt{-det\, g}$$

7.2.5 Cauchy surfaces

The general integral expression for the preserved quantity for an arbitrary Cauchy surface M and for sections $\phi(x^\mu) = (x^\mu, \varphi(x^\mu), \frac{\partial\varphi}{\partial x^\mu}(x^\mu))$ solutions of the Euler-Lagrange equations, and verifying the boundary condition, is given by

$$\int_M \sqrt{-det\, g}\left[g^{\mu\gamma}\frac{\partial\varphi}{\partial x^\mu}\xi^\nu\frac{\partial\varphi}{\partial x^\nu} + g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\xi^\gamma\frac{\partial\varphi}{\partial x^\nu} - \frac{\xi^\gamma}{2}\left(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - m^2\varphi^2\right)\right] d^3x_\gamma$$

In the particular case in which we have M to be a space-like Cauchy surface, g induces a positive definite metric g_M on M, and we have that the preserved quantity is expressed as

$$\int_M \sqrt{-det\, g}\left[\frac{\partial\varphi}{\partial x^0}\xi^\nu\frac{\partial\varphi}{\partial x^\nu} + g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\xi^0\frac{\partial\varphi}{\partial x^\nu} - \frac{\xi^0}{2}\left(g^{\mu\nu}\frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi}{\partial x^\nu} - m^2\varphi^2\right)\right] d^3x_0$$

Whenever ξ_X is space-like (that is, parallel to M), we obtain that the preserved quantity gets

$$\int_M \left[\frac{\partial\varphi}{\partial x^0}\frac{\partial\varphi}{\partial x^\nu}\xi^\nu\right] d^3x_0$$

which is the angular momentum whenever ξ_X is an infinitesimal rotation, and linear momentum whenever it is an infinitesimal translation.

For the contrary, if $\xi_X = \frac{\partial}{\partial x^0}$ we get

$$\frac{1}{2}\int_M \left[\frac{\partial\varphi}{\partial x^0}\frac{\partial\varphi}{\partial x^0} + g^{AB}\frac{\partial\varphi}{\partial x^A}\frac{\partial\varphi}{\partial x^B} + m^2\varphi^2\right] d^3x_0$$

which is the energy of the field φ on the Cauchy surface M.

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